Lectures of

Engineering Analysis

Third Stage

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1- Differential Equation

A ***differential equation*** is an equation involving an unknown function and its derivatives. A differential equation is an ***ordinary differential equation*** (ODE) if the unknown function depends on only one independent variable. If the unknown function depends on two or more independent variables, the differential equation is a ***partial differential equation*** (PDE).

Order : is the highest derivative in the D.E. ***Degree*** : is the highest exponent of an order.

Example: The following are differential equations involving the unknown function y.

$$\frac{dy}{dx} = 9x - 4$$

$$e^{y}\frac{d^{2}y}{dx^{2}} + 2\left(\frac{dy}{dx}\right)^{2} = 1$$

$$4\frac{d^{3}y}{dx^{3}} + \sin x\left(\frac{dy}{dx}\right)^{7} + y^{3}\left(\frac{dy}{dx}\right)^{2} = 5x$$

$$\frac{\partial^{2}y}{\partial t^{2}} - 4\frac{\partial^{2}y}{\partial x^{2}} = 0$$

Expression		Used to represent
y'	$\frac{dy}{dx}$	First derivative
y''	$\frac{d^{2}y}{dx^{2}}$	Second derivative
y'''	$\frac{d^{3}y}{dx^{3}}$	Third derivative
$\mathbf{y^{(4)}}$	$\frac{d^{4}y}{dx^{4}}$	Forth derivative
$\mathbf{y^{(n)}}$	$\frac{d^{n}y}{dx^{n}}$	*n*th derivative
$\mathbf{(y')^{m}}$	$\left(\frac{dy}{dx}\right)^{m}$	(First derivative)^{m}
$\mathbf{(y^{(5)})^{9}}$	$\left(\frac{d^{5}y}{dx^{5}}\right)^{9}$	(Fifth derivative)^{9}
$\mathbf{(y^{(n)})^{m}}$	$\left(\frac{d^{n}y}{dx^{n}}\right)^{m}$	(*n*th derivative)^{m}
$\mathbf{(y^{9})^{(5)}}$	$\frac{d^{5}}{dx^{5}}\left(y^{9}\right)$	Fifth derivative (function)^{m}
$\mathbf{(y^{m})^{(n)}}$	$\frac{d^{n}}{dx^{n}}\left(y^{m}\right)$	*n*th derivative (function)^{m}

2- First Order Differential Equations

Standard form for a First-order differential equation in the unknown function y(x) is:

$$y' = f(x, y)$$

First Order Differential Equation Types:

1. Separable Equations
2. Homogeneous Equations
3. Exact Equations
4. Linear Equations
5. Bernoulli Equations

2.1- Separable Differential Equations

Procedure to solve a Separable First Order Differential Equation

1- Write the equation in the form :

$$\boldsymbol{A(x)dx + B(y)dy = 0}$$

2- Integrate A(*x*) with respect to ***x*** and B(*y*) with respect to ***y*** to obtain an equation that relates y and x .

Example 1 : Solve differential equations

(a) $y(1+x^2)dy = dx$

(b) $dx + xydy + y^2dx + ydy = 0$

(c) $\frac{1}{y}dx - \frac{x}{y^2}dy = 0$

(d) $\frac{dy}{dx} = e^{3x+2y} - 4$

(e) $e^{x+2y}dy - e^{y-2x}dx = 0$

(f) $(y \ln y)dx + (1+x^2)dy = 0 \quad y(0) = e$

Solution : (a) $y(1+x^2)dy = dx$

$$y(1+x^2)dy = dx$$

$$ydy = \frac{dx}{1+x^2}$$

$$\frac{dx}{1+x^2} - ydy = 0$$

$$\int \frac{1}{1+x^2}dx - \int ydy = C \quad \Rightarrow \quad \tan^{-1}x - \frac{y^2}{2} = C \quad \Leftarrow \textbf{Ans.}$$

Solution : (b) $dx + xydy + y^2dx + ydy = 0$

Solution : (c) $\frac{1}{y}dx - \frac{x}{y^2}dy = 0$

$\frac{1}{y}dx - \frac{x}{y^2}dy = 0 \quad \Rightarrow \quad \frac{1}{y}dx = \frac{x}{y^2}dy \quad \Rightarrow \quad \frac{1}{x}dx = \frac{y}{y^2}dy$

$\frac{1}{x}dx - \frac{1}{y}dy = 0$

$\int \frac{1}{x}dx - \int \frac{1}{y}dy = C \quad \Rightarrow \quad \ln x - \ln y = C$

$y = C_1 x$ $\Leftarrow$ **Ans.**

Solution : (d) $\frac{dy}{dx} = e^{3x+2y} + 4$

Solution : (e) $e^{x+2y}dy - e^{y-2x}dx = 0$

$$e^{x+2y}dy - e^{y-2x}dx = 0$$

$$\frac{dy}{dx} = \frac{e^{y-2x}}{e^{x+2y}}$$

$$\frac{dy}{dx} = e^{(y-2x)-(x+2y)} \quad \Rightarrow \quad \frac{dy}{dx} = e^{(y-2y)-(x+2x)}$$

$$\frac{dy}{dx} = e^{-y-3x}e^{y} \quad \Rightarrow \quad dy = e^{-3x}dx$$

$$-e^{-3x}dx + e^{y}dy = 0$$

$$\int -e^{-3x}dx + \int e^{y}dy = C$$

$$\frac{e^{-3x}}{3} + e^{y} = C$$

$$y = ln\left(C - \frac{e^{-3x}}{3}\right)$$ $\Leftarrow$ **Ans.**

Solution : (f) $(y\ ln\ y)dx + (1+x^2)dy = 0 \quad y(0) = e$

$$(y\ ln\ y)dx + (1+x^2)dy = 0$$

$$(y\ ln\ y)dx = -(1+x^2)dy \quad \Rightarrow \quad \frac{1}{(1+x^2)}dx = \frac{-1}{(y\ ln\ y)}dy$$

$$\frac{1}{(1+x^2)}dx + \frac{1}{(y\ ln\ y)}dy = 0$$

$$\int \frac{1}{(1+x^2)}dx + \int \frac{1}{(y\ ln\ y)}dy = C \quad \Rightarrow \quad tan^{-1}(x) + ln(ln\ y) = C$$

$$y(0) = e \quad \Rightarrow \quad tan^{-1}(0) + ln(ln\,e) = C$$

$$0 + ln(1) = C \quad \Rightarrow \quad C = 0$$

$$tan^{-1}(x) + ln(ln\ y) = 0$$

$$y = e^{e^{-tan^{-1}(x)}}$$ $\Leftarrow$ **Ans.**

H.W 1 : Solve differential equations

(a) $\frac{dy}{dx} = \frac{e^{2x+y}}{e^{x-y}}$ (b) $\sqrt{1+(y')^2} = ky \quad (k \text{ is constant})$

(c) $x^2y\frac{dy}{dx} = (1+x)\csc y$ (d) $xe^{y}dy + \frac{1+x^2}{y}dx = 0$

(e) $x(2y-3)dx + (1+x^2)dy = 0$

2.2- Homogenous Differential Equations

Certain first order differential equations are not of the 'variable-separable' type, but can be made separable by changing the variable.

An equation of the form

$$N(x,y)\frac{dy}{dx}=M(x,y)$$

Where N and M are functions of both x and y of the ***same degree*** throughout, is said to be ***Homogeneous*** in y and x.

Homogenous First Order Differential Equation

A first order differential equation is ***Homogenous*** if it can be put in the form:

$$\frac{dy}{dx}=F\left(\frac{y}{x}\right) \qquad (2)$$

or

$$\frac{dx}{dy}=F\left(\frac{x}{y}\right) \qquad (3)$$

$$Let\ v=\frac{y}{x} \quad \Rightarrow \quad y=xv$$

$$\therefore \frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$\because \frac{dy}{dx}=F\left(\frac{y}{x}\right)=F(v)$$

$$\therefore v+x\frac{dv}{dx}=F(v)$$

$$\therefore \frac{dx}{x}+\frac{dv}{v-F(v)}=0$$

$$\therefore \int\frac{dx}{x}+\int\frac{dv}{v-F(v)}=C$$

$$\boxed{\int\frac{dv}{v-F(v)}=C-ln|x|}$$

$$Let\ u=\frac{x}{y} \quad \Rightarrow \quad x=yu$$

$$\therefore \frac{dx}{dy}=u+y\frac{du}{dy}$$

$$\because \frac{dx}{dy}=F\left(\frac{x}{y}\right)=F(u)$$

$$\therefore u+y\frac{du}{dy}=F(u)$$

$$\therefore \frac{dy}{y}+\frac{du}{u-F(u)}=0$$

$$\therefore \int\frac{dy}{y}+\int\frac{du}{u-F(u)}=C$$

$$\boxed{\int\frac{du}{u-F(u)}=C-ln|y|}$$

Procedure to solve a Homogenous First Order Differential Equation

1. Rewrite $N(x,y)\frac{dy}{dx}=M(x,y)$ into the form $\frac{dy}{dx}=\frac{M(x,y)}{N(x,y)}$
2. Make the substitution $v=\frac{y}{x}$.
3. Rewrite $\frac{M(x,y)}{N(x,y)}$ into the form $F\left(\frac{y}{x}\right)=F(v)$.
4. Solve $\int\frac{dv}{v-F(v)}=C-ln|x|$
5. Rewrite the solution into the form of *x* and *y* $(v=\frac{y}{x})$.

<u>Note</u>:

If the form of a Homogenous (F-ODE) is $\frac{dx}{dy}=\frac{M(x,y)}{N(x,y)}$, the steps are:

1. Rewrite $N(x,y)\frac{dy}{dx}=M(x,y)$ into the form $\frac{dx}{dy}=\frac{M(x,y)}{N(x,y)}$
2. Make the substitution $u=\frac{x}{y}$.
3. Rewrite $\frac{M(x,y)}{N(x,y)}$ into the form $F\left(\frac{x}{y}\right)=F(u)$.
4. Solve $\int\frac{du}{u-F(u)}=C-ln|y|$
5. Rewrite the solution into the form of *x* and *y* $(u=\frac{x}{y})$.

Example 2 : Solve differential equations

(a) $\frac{dx}{x-y}=\frac{dy}{x+y}$ (b) $\frac{dx}{xy}=\frac{dy}{x^2+y^2}$

(c) $\frac{dy}{xy}=\frac{dx}{x^2+y^2}$ (d) $\left(xe^{\frac{y}{x}}+y\right)dx-xdy=0$

(e) $\left(x\sin\left(\frac{y}{x}\right)-y\cos\left(\frac{y}{x}\right)\right)dx+\left(x\cos\left(\frac{y}{x}\right)\right)dy=0 \quad , y(2)=\pi$

(f) $\frac{dy}{dx}=\frac{4x+6y+1}{2x-3y}$ (g) $\frac{dy}{dx}=\frac{x+y-2}{x-y+3}$

Solution : (a) $\dfrac{dx}{x-y}=\dfrac{dy}{x+y}$

Solution : (b) $\dfrac{dx}{xy}=\dfrac{dy}{x^2+y^2}$

$$\frac{dx}{xy}=\frac{dy}{x^2+y^2}$$

$$\frac{x^2+y^2}{xy}=\frac{dy}{dx} \quad\Rightarrow\quad \frac{dy}{dx}=\frac{x^2}{xy}+\frac{y^2}{xy}$$

$$\frac{dy}{dx}=\frac{x}{y}+\frac{y}{x}$$

$$Let \quad v=\frac{y}{x} \Rightarrow F(v)=\frac{1}{v}+v$$

$$\because \int\frac{dx}{x}+\int\frac{dv}{v-F(v)}=C \quad\Rightarrow\quad \int\frac{dx}{x}+\int\frac{dv}{v-\left(\frac{1}{v}+v\right)}=C$$

$$\int\frac{dx}{x}-\int v\,dv=C \quad\Rightarrow\quad ln\,x-\frac{v^2}{2}=C \quad\Rightarrow\quad ln\,x-\frac{\left(\frac{y}{x}\right)^2}{2}=C$$

$$y=\pm x\sqrt{2\,ln\,x+C_1} \quad (C_1=2C)$$ $\Leftarrow$ **Ans.**

Solution : (c) $\frac{dy}{xy}=\frac{dx}{x^2+y^2}$

$$\frac{dy}{xy}=\frac{dx}{x^2+y^2}$$

$$\frac{x^2+y^2}{xy}=\frac{dx}{dy} \quad\Rightarrow\quad \frac{dx}{dy}=\frac{x^2}{xy}+\frac{y^2}{xy}$$

$$\frac{dx}{dy}=\frac{x}{y}+\frac{y}{x}$$

$$Let \quad u=\frac{y}{x} \Rightarrow F(u)=\frac{1}{u}+u$$

$$\because \int\frac{dy}{y}+\int\frac{du}{u-F(u)}=C \quad\Rightarrow\quad \int\frac{dy}{y}+\int\frac{du}{u-\left(\frac{1}{u}+u\right)}=C$$

$$\int\frac{dy}{y}-\int u\,du=C \quad\Rightarrow\quad \ln y-\frac{u^2}{2}=C \quad\Rightarrow\quad \ln y-\frac{\left(\frac{x}{y}\right)^2}{2}=C$$

$$2\ln y-\left(\frac{x}{y}\right)^2=2C$$ $\Leftarrow$ **Ans.**

Solution : (d) $\left(xe^{\frac{y}{x}}+y\right)dx-xdy=0$

$$\left(xe^{\frac{y}{x}}+y\right)dx-xdy=0 \quad\Rightarrow\quad \frac{dy}{dx}=\frac{xe^{\frac{y}{x}}+y}{x} \quad\Rightarrow\quad \frac{dy}{dx}=\frac{xe^{\frac{y}{x}}}{x}+\frac{y}{x} \quad\Rightarrow\quad \frac{dy}{dx}=e^{\frac{y}{x}}+\frac{y}{x}$$

$$Let \quad v=\frac{y}{x} \Rightarrow F(v)=e^{v}+v$$

$$\because \int\frac{dx}{x}+\int\frac{dv}{v-F(v)}=C \quad\Rightarrow\quad \int\frac{dx}{x}+\int\frac{dv}{v-(e^{v}+v)}=C$$

$$\therefore \int\frac{dx}{x}-\int e^{-v}dv=C \quad\Rightarrow\quad \ln(x)+e^{-v}=C \quad\Rightarrow\quad \ln(x)+e^{-\frac{y}{x}}=C$$

$$y=-x\ln\left(C-\ln(x)\right)$$ $\Leftarrow$ **Ans.**

Solution : (e) $\left(x\sin\left(\frac{y}{x}\right)-y\cos\left(\frac{y}{x}\right)\right)dx+\left(x\cos\left(\frac{y}{x}\right)\right)dy=0 \quad , y(2)=\pi$

$$\left(\frac{x\sin\left(\frac{y}{x}\right)}{x\cos\left(\frac{y}{x}\right)}-\frac{y\cos\left(\frac{y}{x}\right)}{x\cos\left(\frac{y}{x}\right)}\right)dx+\left(\frac{x\cos\left(\frac{y}{x}\right)}{x\cos\left(\frac{y}{x}\right)}\right)dy=0\div x\cos\left(\frac{y}{x}\right)$$

$$\left(\tan\left(\frac{y}{x}\right)-\frac{y}{x}\right)dx+dy=0 \quad \Rightarrow \frac{dy}{dx}=\frac{y}{x}+\tan\left(\frac{y}{x}\right)$$

$$Let \quad v=\frac{y}{x} \Rightarrow F(v)=v+\tan(v)$$

$$\because \int\frac{dx}{x}+\int\frac{dv}{v-F(v)}=C \quad \Rightarrow \quad \int\frac{dx}{x}+\int\frac{dv}{v-(v+\tan(v))}=C$$

$$\int\frac{dx}{x}-\int\cot(v)dv=C \quad \Rightarrow \quad \ln x-\ln(\sin(v))=C \quad \Rightarrow \quad y=x\sin^{-1}\left(\frac{x}{C}\right)$$

$$y(2)=\pi \quad \Rightarrow \quad \pi=2\sin^{-1}\left(\frac{2}{C}\right) \quad \Rightarrow \quad C=2$$

$$y=x\sin^{-1}\left(\frac{x}{2}\right)$$ $\Leftarrow$ **Ans.**

Solution : (f) $\frac{dy}{dx}=\frac{4x+6y+1}{2x-3y}$

Solution : (g) $\dfrac{dy}{dx}=\dfrac{x+y-2}{x-y+3}$

H.W 2 : Solve differential equations

(a) $\dfrac{dy}{dx}=\dfrac{\sqrt{x^2+y^2}-x}{y}$

(b) $x\dfrac{dy}{dx}=y+\sqrt{x^2+y^2}$

(c) $\left(x^2+y^2\right)dy-y^2dx=0$

(d) $\dfrac{dy}{dx}=\dfrac{e^y-e^x}{e^y+e^x}+1$

(e) $\dfrac{dy}{dx}=\dfrac{2x-y}{x-2y}$

(f) $x\dfrac{dy}{dx}-y=\sqrt{x^2-y^2}$

(g) $\dfrac{dy}{dx}=\dfrac{x+y}{x-y+2}$

(h) $\dfrac{dy}{dx}=\dfrac{5x-3y+7}{5x-3y+1}$

(i) $y'=\dfrac{y\left(1+ln(y)-ln(x)\right)}{x\left(ln(y)-ln(x)\right)}$, $y(1)=1$

2.3- Exact Differential Equations

A first order differential equation is ***Exact*** if it can be put in the form:

$$M(x,y)dx + N(x,y)dy = 0 \quad (4)$$

or

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \quad (5)$$

$$\therefore M = \frac{\partial f}{\partial x} \quad , \quad \therefore N = \frac{\partial f}{\partial y}$$

And having the property that

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Procedure to solve a Exact Differential Equation

1. Rewrite the equation in the form :

$$M(x,y)dx + N(x,y)dy = 0$$

2. Integrate $M(x,y)$ with respect to x, writing the constant of integration as $k(y)$

$$f(x,y) = \int_{y\ constant} M(x,y)\,dx + k(y) \quad (6)$$

3. Differentiate with respect to y, and set result equal $N(x,y)$ to find $k'(y)$

$$N(x,y) = \frac{1}{\partial y}\left(\int_{y\ constant} M(x,y)\,dx\right) + k'(y)$$

4. Integrate to find $k(y)$ and substituted in Eq.(6), then writing the solution of exact equation as

<u>General Solution</u>

$$R = \int_{y-constant} M(x,y)\,dx \qquad K = \int_{x-constant}\left(N - \frac{\partial R}{\partial y}\right)dy = \int N(0,y)\,dy$$

$$\boxed{R + K = C}$$

Example 3 : Solve differential equations

(a) $2xydx+\left(1+x^2\right)dy=0$

(b) $\left(y^2-1\right)dx+\left(\sin y-2xy\right)dy=0$

(c) $\left(e^x+\ln(y)+\frac{y}{x}\right)dx+\left(\frac{x}{y}+\ln(x)+\sin(y)\right)dy=0$

(d) $\left(x+\sqrt{1+y^2}\right)dx-\left(y-\frac{xy}{\sqrt{1+y^2}}\right)dy=0$

Solution : (a) $2xydx+\left(1+x^2\right)dy=0$

Solution : (b) $\left(y^2-1\right)dx+\left(\sin y-2xy\right)dy=0$

Solution : (c) $\left(e^{x}+\ln(y)+\frac{y}{x}\right)dx+\left(\frac{x}{y}+\ln(x)+\sin(y)\right)dy=0$

$$M=e^{x}+\ln(y)+\frac{y}{x} \quad \& \quad N=\frac{x}{y}+\ln(x)+\sin(y)$$

$$\frac{\partial M}{\partial y}=\frac{1}{y}+\frac{1}{x} \quad \& \quad \frac{\partial N}{\partial x}=\frac{1}{y}+\frac{1}{x} \quad (\therefore exact)$$

$$f(x,y)=\int_{y\ constant}\left(e^{x}+\ln(y)+\frac{y}{x}\right)dx+k(y)$$

$$f(x,y)=e^{x}+x\ln(y)+y\ln(x)+k(y)$$

$$N=\frac{\partial f}{\partial y} \Rightarrow \frac{x}{y}+\ln(x)+\sin(y)=\frac{x}{y}+\ln(x)+k'(y) \Rightarrow k'(y)=\sin(y)$$

$$\therefore k(y)=-\cos(y)$$

$$\therefore e^{x}+x\ln(y)+y\ln(x)-\cos(y)=C \qquad \Leftarrow \textbf{Ans.}$$

Solution : (d) $\left(x+\sqrt{1+y^{2}}\right)dx-\left(y-\frac{xy}{\sqrt{1+y^{2}}}\right)dy=0$

$$M=x+\sqrt{1+y^{2}} \quad \& \quad N=y-\frac{xy}{\sqrt{1+y^{2}}}$$

$$\frac{\partial M}{\partial y}=-\frac{y}{\sqrt{1+y^{2}}} \quad \& \quad \frac{\partial N}{\partial x}=-\frac{y}{\sqrt{1+y^{2}}} \quad (\therefore Exact)$$

$$R=\int_{y-constant}M\,dx=\int_{y-constant}\left(x+\sqrt{1+y^{2}}\right)dx=\frac{x^{2}}{2}+x\sqrt{1+y^{2}}$$

$$\frac{\partial R}{\partial y}=-\frac{xy}{\sqrt{1+y^{2}}}$$

$$K=\int_{x-constant}\left(N-\frac{\partial R}{\partial y}\right)dy=\int_{x-constant}\left(y-\frac{xy}{\sqrt{1+y^{2}}}-\left(-\frac{xy}{\sqrt{1+y^{2}}}\right)\right)dy=\int_{x-constant}y\,dy=\frac{y^{2}}{2}$$

$$R+K=C \Rightarrow \frac{x^{2}}{2}+x\sqrt{1+y^{2}}+\frac{y^{2}}{2}=C \qquad \Leftarrow \textbf{Ans.}$$

H.W 3 : Solve differential equation

(a) $\left(e^{x}+\ln(y)\right)dx+\left(\frac{x}{y}+1\right)dy=0\ ,\ y(\ln(2))=1$

(b) $\left(\frac{y^{2}}{1+x^{2}}-2y\right)dx+\left(2y\tan^{-1}(x)-2x+\sinh(y)\right)dy=0$

(c) $\left(x+\sqrt{1+y^{2}}\right)dx-\left(y-\frac{xy}{\sqrt{1+y^{2}}}\right)dy=0$

(d) $\left(\sin(x)+\tan^{-1}\left(\frac{y}{x}\right)\right)dx-\left(y-\ln\left(\sqrt{x^{2}+y^{2}}\right)\right)dy=0$

2.3.1 Integration factor

It can be shown that every ***nonexact*** $\left(\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\right)$ differential equation, can be made exact by multiplying both sides by a suitable ***integrating factor*** $I(x, y)$.

$$I(x,y)M(x,y)dx + I(x,y)N(x,y)dy = 0$$

	Condition	Integrating factor
Case [1]	If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(x)$	$I(x,y) = e^{\int g(x)dx}$
Case [2]	If $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = h(y)$	$I(x,y) = e^{-\int h(y)dy}$
Case [3]	If $M = y\,f(xy)$ and $N = x\,g(xy)$	$I(x,y) = \frac{1}{xM - yN}$

Example 4 : Solve differential equations

(a) $ydx - xdy = 0$ (b) $(x + 3y)dx + xdy = 0$ (c) $2xy^2dx + 3x^2ydy = 0$

Solution : (a) $ydx - xdy = 0$

$M = y$ & $N = -x$ $\frac{\partial M}{\partial y} = 1$ & $\frac{\partial N}{\partial x} = -1$ $(\therefore nonexact)$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(x) = \frac{1}{-x}(1-(-1)) = -\frac{2}{x}$$

$$I = e^{\int g(x)dx} = e^{\int -\frac{2}{x}dx} = \frac{1}{x^2}$$

$$Integtation\ factor\ I = \frac{1}{x^2}$$

$$(IM)dx + (IN)dy = 0 \quad \Rightarrow \quad \left(\frac{y}{x^2}\right)dx + \left(-\frac{1}{x}\right)dy = 0$$

$$\frac{\partial(IM)}{\partial y} = \frac{1}{x^2} \quad \& \quad \frac{\partial(IN)}{\partial x} = \frac{1}{x^2} \quad (\therefore exact)$$

$$R = \int_{y-constant} M^*(x,y)\,dx = \int_{y-constant} \left(\frac{y}{x^2}\right)dx = -\frac{y}{x}$$

$$K = \int N^*(0,y)\,dy = C_1$$

$$R + K = C \quad \Rightarrow \quad -\frac{y}{x} + C_1 = C \quad \Rightarrow \frac{y}{x} = C_2$$ ⇐ **Ans.**

Solution : (b) $(x+3y)dx + xdy = 0$

$$M = x + 3y \quad \& \quad N = x$$

$$\frac{\partial M}{\partial y} = 3 \quad \& \quad \frac{\partial N}{\partial x} = 1 \quad (\therefore nonexact)$$

$$Integtation\ factor\ I = x^2$$

$$(IM)dx + (IN)dy = 0 \quad \Rightarrow \quad (x^3 + 3x^2y)dx + x^3dy = 0$$

$$\frac{\partial (IM)}{\partial y} = 3x^2 \quad \& \quad \frac{\partial (IN)}{\partial x} = 3x^2 \quad (\therefore exact)$$

$$f(x,y) = \int_{y\ constant} (x^3 + 3x^2y)dx + k(y)$$

$$f(x,y) = \frac{x^4}{4} + x^3y + k(y)$$

$$IN = \frac{\partial f}{\partial y} \quad \Rightarrow \quad x^3 = x^3 + k'(y) \quad \Rightarrow \quad k'(y) = 0$$

$$\therefore k(y) = C_1$$

$$\therefore \frac{x^4}{4} + x^3y + C_1 = C \quad \quad \frac{x^4}{4} + x^3y = C_2$$ $\Leftarrow$ **Ans.**

Solution : (c) $2xy^2dx + 3x^2ydy = 0$

$$2xy^2dx + 3x^2ydy = 0$$

$$M = 2xy^2 \quad \& \quad N = 3x^2y$$

$$\frac{\partial M}{\partial y} = 4xy \quad \& \quad \frac{\partial N}{\partial x} = 6xy \quad (\therefore nonexact)$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{3x^2y}(4xy - 6xy) = \frac{-2}{3x} = g(x) \quad \quad I = e^{\int g(x)dx} = \frac{1}{x^{\frac{2}{3}}}$$

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2xy^2}(4xy - 6xy) = \frac{-1}{y} = h(y) \quad \quad I = e^{-\int h(y)dy} = y$$

$$M = y\,f(xy) \quad and \quad N = x\,g(xy) \quad \quad I = \frac{1}{xM - yN} = \frac{-1}{x^2y^2}$$

$$Integtation\ factor\ I = y \quad \quad (IM)dx + (IN)dy = 0 \quad \Rightarrow \quad 2xy^3dx + 3x^2y^2dy = 0$$

$$\frac{\partial (IM)}{\partial y} = 6xy^2 \quad \& \quad \frac{\partial (IN)}{\partial x} = 6xy^2 \quad (\therefore exact)$$

$$R = \int_{y-constant} IM\ dx = \int_{y-constant} (2xy^3)\ dx = x^2y^3 \quad \quad \frac{\partial R}{\partial y} = 3x^2y^2$$

$$K = \int_{x-constant} \left(IN - \frac{\partial R}{\partial y}\right)dy = \int_{x-constant} \left((3x^2y^2) - (3x^2y^2)\right)dy = \int_{x-constant} (0)dy = C_1$$

$$R + K = C \quad \Rightarrow \quad x^2y^3 + C_1 = C \Rightarrow x^2y^3 = C_2$$ $\Leftarrow$ **Ans.**

2.4- Linear First-Order Differential Equations

A differential equation that can be written in the form:

$$\frac{dy}{dx}+P(x)y=Q(x) \tag{7}$$

Where $P(x)$ and $Q(x)$ are functions of x, is called a linear first order equation

Note: We can be written a linear first order equation in the form:

$$\frac{dx}{dy}+P(y)x=Q(y)$$

Procedure to solve a Linear First-Order Differential Equation

1. Rewrite the equation in standard form :

$$\frac{dy}{dx}+P(x)y=Q(x)$$

2. Find $\rho(x)$

$$\rho(x)=e^{\int P(x)dx}$$

3. Find $y(x)$

$$y(x)=\frac{1}{\rho(x)}\int \rho(x)Q(x)dx$$

Example 5 : Solve differential equations

(a) $\frac{dy}{dx}-\frac{1}{x}y=x^2$ (b) $x\frac{dy}{dx}+y=x$

(c) $\cosh(x)\frac{dy}{dx}+\sinh(x)y=e^{-x}$ (d) $(1+x)\frac{dy}{dx}+2y=x \quad , y(0)=1$

Solution : (a) $\frac{dy}{dx}-\frac{1}{x}y=x^2$

Solution : (b) $x\frac{dy}{dx}+y=x$

Solution : (c) $cosh(x)\frac{dy}{dx}+sinh(x)y=e^{-x}$

$$cosh(x)\frac{dy}{dx}+sinh(x)y=e^{-x}$$

$$\left(\frac{cosh(x)}{cosh(x)}\right)\frac{dy}{dx}+\left(\frac{sinh(x)}{cosh(x)}\right)y=\left(\frac{e^{-x}}{cosh(x)}\right) \qquad \div cosh(x)$$

$$\frac{dy}{dx}+tanh(x)y=e^{-x}sech(x)$$

$$P(x)=tanh(x) \qquad \& \qquad Q(x)=e^{-x}sech(x)$$

$$\rho(x)=e^{\int P(x)dx}=e^{\int tanh(x)dx}=e^{ln(cosh(x))}=cosh(x)$$

$$y(x)=\frac{1}{\rho(x)}\int\rho(x)Q(x)dx$$

$$y(x)=\frac{1}{cosh(x)}\int cosh(x)e^{-x}sech(x)dx$$

$$y(x)=\frac{1}{cosh(x)}\int e^{-x}dx=\frac{1}{cosh(x)}\left(-e^{-x}+C\right)=\frac{1}{\left(\frac{e^{x}+e^{-x}}{2}\right)}\left(-e^{-x}+C\right)$$

$$y(x)=\frac{C_1e^{x}-2}{e^{2x}+1}$$ $\Leftarrow$ **Ans.**

Solution : (d) $(1+x)\frac{dy}{dx}+2y=x \quad , y(0)=1$

$$(1+x)\frac{dy}{dx}+2y=x$$

$$\frac{dy}{dx}+\frac{2}{1+x}y=\frac{x}{1+x}$$

$$P(x)=\frac{2}{1+x} \qquad \& \qquad Q(x)=\frac{x}{1+x}$$

$$\rho(x)=e^{\int P(x)dx}=e^{\int\frac{2}{1+x}dx}=e^{2ln(1+x)}=(1+x)^2$$

$$y(x)=\frac{1}{(1+x)^2}\int(1+x)^2\left(\frac{x}{1+x}\right)dx=\frac{1}{(1+x)^2}\int x(1+x)dx=\frac{1}{(1+x)^2}\int\left(x+x^2\right)dx$$

$$y(x)=\frac{\left(\frac{x^2}{2}+\frac{x^3}{3}\right)+C}{(1+x)^2} \quad \Rightarrow \quad 1=\frac{\left(\frac{0^2}{2}+\frac{0^3}{3}\right)+C}{(1+0)^2} \quad \Rightarrow \quad C=0$$

$$y(x)=\left(\frac{x}{1+x}\right)^2\left(\frac{1}{2}+\frac{x}{3}\right)$$ $\Leftarrow$ **Ans.**

H.W 5: Solve differential equations

(a) $x\frac{dy}{dx}+3y=\frac{sin(x)}{x^2}$

(b) $e^{2y}dx+2\left(xe^{2y}-y\right)dy=0$

(c) $\left(1+y^2\right)dx+(1+2xy)dy=0$

(d) $(x-1)^3\frac{dy}{dx}+4(x-1)^2y=x+1$

(e) $x\frac{d^2y}{dx^2}+\frac{dy}{dx}=x^2$

(f) $x\frac{dy}{dx}+2y=1+x^2 \ , y(1)=1$

2.5- Bernoulli Equations

A Bernoulli differential equation has the form:

$$y' + P(x)y = Q(x)y^n \tag{8}$$

Where *n* is a real number. The substitution

$$z = y^{1-n}$$

$$y' + P(x)y = Q(x)y^n$$

$$\frac{y'}{y^n} + P(x)\frac{1}{y^{n-1}} = Q(x)$$

$$Let \qquad z = \frac{1}{y^{n-1}} \qquad \frac{dz}{dx} = (1-n)\frac{y'}{y^n} \qquad \therefore \frac{y'}{y^n} = \frac{1}{(1-n)}\frac{dz}{dx}$$

$$\frac{1}{(1-n)}\frac{dz}{dx} + P(x)z = Q(x)$$

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)$$

Procedure to solve a Bernoulli Differential Equation

1. Rewrite the equation in standard form :

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)$$

$$\frac{dz}{dx} + P^*(x)z = Q^*(x)$$

$$Where \qquad P^*(x) = (1-n)P(x) \quad , \quad Q^*(x) = (1-n)Q(x)$$

2. Find $\rho(x)$

$$\rho(x) = e^{\int P^*(x)dx}$$

3. Find $y(x)$

$$z(x) = \frac{1}{\rho(x)}\int \rho(x)Q^*(x)dx$$

Then

$$(y(x))^{(1-n)} = \frac{1}{\rho(x)}\int \rho(x)Q^*(x)dx$$

Example 6 : Solve differential equations

(a) $y\dfrac{dy}{dx}+y^{2}=x$ (b) $\dfrac{dy}{dx}+\dfrac{y}{x}=y^{2}$ (c) $\dfrac{dy}{dx}-\dfrac{1}{x}y=-\dfrac{\cos x}{x^{3}}y^{4}$

Solution : (a) $y\dfrac{dy}{dx}+y^{2}=x$

Solution : (b) $\dfrac{dy}{dx}+\dfrac{y}{x}=y^{2}$

Solution : (c) $\frac{dy}{dx}-\frac{1}{x}y=-\frac{\cos x}{x^3}y^4$

$$\frac{dy}{dx}-\frac{1}{x}y=-\frac{\cos x}{x^3}y^4$$

$$\frac{y'}{y^4}+\left(-\frac{1}{x}\right)\frac{1}{y^3}=-\frac{\cos x}{x^3}$$

$$n=4 \qquad Let \quad z=\frac{1}{y^3}$$

$$\frac{dz}{dx}+P^*(x)z=Q^*(x)$$

$$\frac{dz}{dx}+(1-4)\left(-\frac{1}{x}\right)z=(1-4)\left(-\frac{\cos x}{x^3}\right) \qquad \frac{dz}{dx}+\left(\frac{3}{x}\right)z=\left(\frac{3\cos x}{x^3}\right)$$

$$P^*(x)=\frac{3}{x}$$

$$Q^*(x)=\frac{3\cos x}{x^3}$$

$$\rho(x)=e^{\int P^*(x)dx}=e^{\int\left(\frac{3}{x}\right)dx}=x^3$$

$$z(x)=\frac{1}{\rho(x)}\int \rho(x)Q^*(x)dx$$

$$z(x)=\frac{1}{x^3}\int x^3\left(\frac{3\cos x}{x^3}\right)dx=\frac{1}{x^3}(3\sin x+C)$$

$$\frac{1}{y^3}=\frac{1}{x^3}(3\sin x+C)$$

$$\frac{x^3}{y^3}=3\sin x+C$$ $\Leftarrow$ **Ans.**

H.W 6: Solve differential equations

(a) $2y-3\frac{dy}{dx}=y^4e^{3x}$

(b) $xdy+(3y-x^3y^2)dx=0$

(c) $(x-2y)dy+ydx=0$

(d) $(\sin^2 x-y)dx-\tan xdy=0$

(e) $(x^2+x)dy=(x^5+3xy+3y)dx$

(f) $y''+2y'=4x$

(g) $4y(y')^2y''=(y')^4+3$

3- Second Order Linear Homogeneous Equations

The linear equation

$$\frac{d^2y}{dx^2}+2a\frac{dy}{dx}+by=0 \qquad \Rightarrow \qquad \left(D^2+2aD+b\right)y=0$$

is called ***second order linear homogenous equation***

D is called a ***linear differential operator***

The equation

$$r^2+2ar+b=0$$

is the ***characteristic equation*** of the equation

$$\frac{d^2y}{dx^2}+2a\frac{dy}{dx}+by=0$$

Solution of $\frac{d^2y}{dx^2}+2a\frac{dy}{dx}+by=0$ **Roots of** $r^2+2ar+b=0$	**Solution**
r_1, r_2 **real and unequal**	$y=C_1e^{r_1x}+C_2e^{r_2x}$
r_1, r_2 **real and equal**	$y=\left(C_1x+C_2\right)e^{r_2x}$
r_1, r_2 **complex conjugate ,** $\alpha\pm\beta i$	$y=e^{\alpha x}\left(C_1\cos\beta x+C_2\sin\beta x\right)$

Example 11 : Solve differential equation

(a) $y''+2y'=0$ (b) $y''-4y'+4y=0$ (c) $y''-2y'+4y=0$

Solution : (a) $y''+2y'=0$

$$y''+2y'=0$$
$$\left(D^2+2D\right)y=0$$
$$\left(r^2+2r\right)=0$$
$$r\left(r+2\right)=0$$
$$\therefore r_1=0 \ \& \ r_2=-2 \qquad \left(r_1 \,\&\, r_2 \ \textit{real and unequal}\right)$$
$$y=C_1e^{0x}+C_2e^{-2x} \qquad \Rightarrow \qquad y=C_1+C_2e^{-2x} \qquad \Leftarrow \textbf{Ans.}$$

Solution : (b) $y''-4y'+4y=0$

$$y''-4y'+4y=0$$
$$\left(D^2-4D+4\right)y=0$$
$$\left(r^2-4r+4\right)=0$$
$$\left(r-2\right)^2=0$$
$$\therefore r_1=r_2=2 \qquad \left(r_1 \,\&\, r_2 \ \textit{real and equal}\right)$$
$$y=\left(C_1x+C_2\right)e^{r_2x} \quad \Rightarrow \quad y=C_1+C_2e^{2x} \qquad \Leftarrow \textbf{Ans.}$$

Solution : (c) $y''-2y'+4y=0$

$$y''-2y'+4y=0$$
$$\left(D^2-2D+4\right)y=0$$
$$\left(r^2-2r+4\right)=0$$
$$r=\frac{-(-2)\pm\sqrt{(-2)^2-4\times1\times4}}{2\times1}$$
$$\therefore r_1=1\pm\sqrt{3}\,i \qquad \left(r_1\,\&\,r_2\ \ complex\ conjugate\right)$$
$$\alpha=1\ ,\ \beta=\sqrt{3}$$
$$y=e^{\alpha x}\left(C_1\cos\beta x+C_2\sin\beta x\right) \quad\Rightarrow\quad y=e^{x}\left(C_1\cos\sqrt{3}x+C_2\sin\sqrt{3}x\right) \qquad \Leftarrow \textbf{Ans.}$$

H.W 5 : Solve differential equation

(a) $y''-9y=0 \qquad y(\ln 2)=1\ ,\ y'(\ln 2)=-3$

(b) $4y''+12y'+9y=0 \qquad y(0)=0\ ,\ y'(0)=1$

(c) $y''-6y'+10y=0 \qquad y(0)=7\ ,\ y'(0)=1$

4- Second Order Linear Nonhomogeneous DE

If, the linear equation $F(x) \neq 0$

$$\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + by = F(x)$$

is called ***second order linear nonhomogeneous equation***

The general solution of the nonhomogeneous equation is

$$y = y_h + y_p$$

$Where$

$y_h = homogeneous\ \ solution$

$y_p = particular\ solution$

4.1- Method of Variation of Parameters

Steps of solution second order linear nonhomogeneous equation by variation of parameters method:

Procedure to use a Method of Variation of Parameters

1. Find $y_h = C_1\, y_1(x) + C_2\, y_2(x)$
2. Calculate

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = -y_2\, f(x)$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = +y_1\, f(x)$$

3. Find u_1 and u_2

$$u_1 = \int \frac{W_1}{W}\, dx \quad , \quad u_2 = \int \frac{W_2}{W}\, dx$$

4. Write the particular solution as $\left[y_p = u_1 y_1 + u_2 y_2 \right]$
5. Write the general solution as $\left[y = y_h + y_p \right]$

Example 1 : Solve differential equation

(a) $y''-y=e^x$ (b) $y''-y'=e^x\cos x$ (c) $y''+y=\sec x\tan x$

Solution : (a) $y''-y=e^x$

$$y''-y=e^x$$

First , find y_h

$$\left(D^2-1\right)y=0$$

$$r^2-1=0 \quad\Rightarrow\quad r_1=1 \quad \& \quad r_2=-1$$

$$y=C_1e^x+C_2e^{-x}$$

$$\therefore u_1=e^x \quad , \quad u_2=e^{-x} \quad , \quad F(x)=e^x$$

$$\therefore u_1'=e^x \quad , \quad u_2'=-e^{-x}$$

$$D=\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} \quad\Rightarrow\quad D=\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}=-e^xe^{-x}-e^xe^{-x}=-1-1=-2$$

$$v_1'=\frac{-u_2F(x)}{D}\Rightarrow \quad v_1'=\frac{-e^{-x}e^x}{-2}=\frac{1}{2} \quad \Rightarrow v_1=\frac{x}{2}+C_1$$

$$v_2'=\frac{u_1F(x)}{D}\Rightarrow \quad v_2'=\frac{e^xe^x}{-2}=-\frac{1}{2}e^{2x} \quad \Rightarrow v_2=C_2-\frac{1}{4}e^{2x}$$

$$y=v_1u_1+v_2u_2$$

$$y=\left(\frac{x}{2}+C_1\right)e^x+\left(C_2-\frac{1}{4}e^{2x}\right)e^{-x} \qquad \left(where\ C_3=C_1-\frac{1}{4}\right)$$

$$y=\frac{1}{2}xe^x+C_3e^x+C_2e^{-x}$$ $\Leftarrow$ **Ans.**

Solution : (b) $y''-y'=e^x\cos x$

$$y''-y'=e^x\cos x$$

First , find y_h

$$\left(D^2-D\right)y=0$$

$$r^2-r=0 \quad \Rightarrow r(r-1)=0 \quad \Rightarrow r_1=0 \quad \& \quad r_2=1$$

$$y=C_1+C_2e^x$$

$$\therefore u_1=1 \quad , \quad u_2=e^x \quad , \quad F(x)=e^x\cos x$$

$$\therefore u_1'=0 \quad , \quad u_2'=e^x$$

$$D=\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} \quad\Rightarrow\quad D=\begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix}=e^x-0 \quad \Rightarrow D=e^x$$

$$v_1'=\frac{-u_2F(x)}{D}\Rightarrow \quad v_1'=\frac{-e^x\times e^x\cos x}{e^x}=-e^x\cos x \quad \Rightarrow v_1=C_1-\frac{1}{2}e^x(\cos x+\sin x)$$

$$v_2'=\frac{u_1F(x)}{D} \quad\Rightarrow\quad v_2'=\frac{1\times e^x\cos x}{e^x}=\cos x \quad \Rightarrow v_2=C_2+\sin x$$

$$y=\left(C_1-\frac{1}{2}e^x\left(\cos x+\sin x\right)\right)\times 1+\left(C_2+\sin x\right)\times e^x$$

$$y=C_1+C_2e^x+\frac{1}{2}e^x\left(\cos x-\sin x\right)$$ $\Leftarrow$ **Ans.**

Solution : (c) $y''+y=\sec x\ \tan x$

$y''+y=\sec x\ \tan x$

First , find y_h

$y''+y=0$

$(D^2+1)y=0$

$r^2+1=0 \quad\Rightarrow\quad r=\pm i \quad (\alpha=0,\beta=0)$

$y=C_1\cos x+C_2\sin x$

$\therefore u_1=\cos x \quad , \quad u_2=\sin x \quad , \quad F(x)=\sec x\ \tan x$

$\therefore u_1'=-\sin x \quad , \quad u_2'=\cos x$

$$D=\begin{vmatrix}u_1 & u_2\\ u_1' & u_2'\end{vmatrix} \Rightarrow D=\begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix}=\cos^2 x-\left(-\sin^2 x\right) \quad\Rightarrow\quad D=1$$

$$v_1'=\frac{-u_2\ F(x)}{D}\Rightarrow \quad v_1'=\frac{-\sin x\times\sec x\ \tan x}{1}=-\tan^2 x \quad\Rightarrow v_1=x-\tan x+C_1$$

$$v_2'=\frac{u_1\ F(x)}{D}\Rightarrow \quad v_2'=\frac{\cos x\times\sec x\ \tan x}{1}=\tan x \quad\Rightarrow v_2=\ln\left|\sec x\right|+C_2$$

$$y=\left(x-\tan x+C_1\right)\times\cos x+\left(\ln\left|\sec x\right|+C_2\right)\times\sin x \qquad (C_3=C_2-1)$$

$$y=x\cos x+\sin x\ \ln\left|\sec x\right|+C_1\cos x+C_3\sin x$$ $\Leftarrow$ **Ans.**

H.W 6 : Solve differential equation

(a) $y''+y=\csc x$

(b) $y''+y=\cot x$

(c) $y''-y'=e^x+e^{-x}$

(d) $y''-4y'-5y=e^x+4$

(e) $y''+y=\sec^2 x$

4.2- Method of Undetermined Coefficients

Procedure to use a Method of Undetermined Coefficients

2. Find y_h
3. Find y_p
4. Write the general solution as $\left[y = y_h + y_p\right]$

$F(x)$	Condition	y_p
e^{kx}	$k \neq r_1$ and $k \neq r_2$	Ae^{kx}
	$k = r_1$ or $k = r_2$	Axe^{kx}
	$k = r_1$ and $k = r_2$	Ax^2e^{kx}
$\sin kx, \cos kx$	$k \neq r_1$ and $k \neq r_2$	$B\cos kx + C\sin kx$
	$k = r_1$ and $k = r_2$	$Bx\cos kx + Cx\sin kx$
$ax^2 + bx + c$	$0 \neq r_1$ and $0 \neq r_2$	$Dx^2 + Ex + F$
	$0 = r_1$ or $0 = r_2$	$Dx^3 + Ex^2 + Fx$
	$0 = r_1$ and $0 = r_2$	$Dx^4 + Ex^3 + Fx^2$

Example 7 : Solve differential equation

(a) $y'' - y = e^x + x^2$ (b) $y'' - y' - 6y = e^{-x} - 7\cos x$

Solution : (a) $y'' - y = e^x + x^2$

$$y'' - y = e^x + x^2$$

First, find y_h

$$y'' - y = 0$$

$$\left(D^2 - 1\right)y = 0$$

$$r^2 - 1 = 0 \quad \Rightarrow \quad r_1 = 1 \quad \& \quad r_2 = -1$$

$$y = C_1 e^x + C_2 e^{-x}$$

Second, find y_p

$$\because F(x)_1 = e^x \ , \ k = 1 = r_1 \quad \Rightarrow \quad \therefore \left(y_p\right)_1 = Axe^x$$

$$\because F(x)_2 = x^2 \ , \ r_1 \,\&\, r_2 \neq 0 \quad \Rightarrow \quad \therefore \left(y_p\right)_2 = Bx^2 + Cx + D$$

$$y_p = (y_p)_1 + (y_p)_2$$

$$y_p = Axe^x + Bx^2 + Cx + D$$

$$y_p' = A(xe^x + e^x) + 2Bx + C$$

$$y_p'' = A(xe^x + 2e^x) + 2B = Axe^x + 2Ae^x + 2B$$

$$y_p'' - y_p = e^x + x^2$$

$$(Axe^x + 2Ae^x + 2B) - (Axe^x + Bx^2 + Cx + D) = e^x + x^2$$

$$2Ae^x - Bx^2 - Cx + 2B - D = e^x + x^2$$

$$\therefore\ 2Ae^x = e^x \quad \Rightarrow \quad A = \frac{1}{2}$$

$$\therefore\ -Bx^2 = x^2 \quad \Rightarrow \quad B = -1$$

$$\therefore\ -Cx = 0x \quad \Rightarrow \quad C = 0$$

$$\therefore\ 2B - D = 0 \quad \Rightarrow \quad D = 2B = -2$$

$$\therefore\ y_p = \frac{1}{2}xe^x - x^2 - 2$$

$$\therefore\ y = y_h + y_p \quad \Rightarrow \quad y = C_1 e^x + C_2 e^{-x} + \frac{1}{2}xe^x - x^2 - 2$$ $\Leftarrow$ **Ans.**

Solution : (b) $y'' - y' - 6y = e^{-x} - 7\cos x$

$$y'' - y' - 6y = e^{-x} - 7\cos x$$

First , find y_h

$$(D^2 - D - 6)y = 0$$

$$r^2 - r - 6 = 0 \quad \Rightarrow \quad (r-3)(r+2) = 0 \quad \Rightarrow \quad r_1 = 3 \quad \& \quad r_2 = -2$$

$$y = C_1 e^{3x} + C_2 e^{-2x}$$

Second , find y_p

$$\because F(x)_1 = e^{-x} \quad , \quad k = -1 \quad (r_1 \,\&\, r_2 \neq k) \quad \Rightarrow \quad \therefore (y_p)_1 = Ae^{-x}$$

$$\because F(x)_2 = \cos x \quad , \quad k = 1 \quad (r_1 \,\&\, r_2 \neq k) \quad \Rightarrow \quad \therefore (y_p)_2 = B\cos x + C\sin x$$

$$y_p = (y_p)_1 + (y_p)_2 \quad \Rightarrow \quad y_p = Ae^{-x} + B\cos x + C\sin x$$

$$y_p' = -Ae^{-x} - B\sin x + C\cos x$$

$$y_p'' = Ae^{-x} - B\cos x - C\sin x$$

$$\because y_p'' - y_p' - 6y_p = e^{-x} - 7\cos x$$

$$(Ae^{-x} - B\cos x - C\sin x) - (-Ae^{-x} - B\sin x + C\cos x) - 6(Ae^{-x} + B\cos x + C\sin x) = e^{-x} - 7\cos x$$

$$(1+1-6)Ae^{-x} + (-B - C - 6B)\cos x + (-C + B - 6C)\sin x = e^{-x} - 7\cos x$$

$\therefore -4Ae^{-x} = e^{-x} \quad \Rightarrow \quad A = -\frac{1}{4}$

$\therefore -(7B + C)\cos x = -7\cos x \quad \Rightarrow \quad 7B + C = 7 \quad .....(1)$

$\therefore (B - 7C)\sin x = 0\sin x \quad \Rightarrow \quad B - 7C = 0 \quad .....(2)$

$\therefore B = \frac{49}{50} \quad , \quad C = \frac{7}{50}$

$\therefore y_p = -\frac{1}{4}e^{-x} + \frac{49}{50}\cos x + \frac{7}{50}\sin x$

$\therefore y = y_h + y_p \quad \Rightarrow \quad y = C_1 e^{3x} + C_2 e^{-2x} - \frac{1}{4}e^{-x} + \frac{49}{50}\cos x + \frac{7}{50}\sin x$ $\Leftarrow$ **Ans.**

Differential Equation	y_p
$y''+2ay'+by=c$	$\frac{c}{b}$
$y''+2ay'=c$	$\frac{c}{2a}x$
$y''+2ay'+by=cx+d$	$\frac{c}{b}x+\frac{bd-2ac}{b^2}$
$y''+2ay'=cx+d$	$\frac{c}{4a}x^2+\frac{2ad-c}{4a^2}x$
$y''=cx+d$	$\frac{c}{6}x^3+\frac{d}{2}x^2$
$y''+2ay'+by=ce^{kx}$	$\frac{c}{k^2+2ak+b}e^{kx}$
$y''+2ay'+by=ce^{kx} \quad k=r_1 \text{ or } r_2$	$\frac{c}{2a+2k}xe^{kx}$
$y''+2ay'+by=ce^{kx} \quad k=r_1=r_2$	$\frac{c}{2}x^2e^{kx}$
$y''+2ay'+by=c\cos kx \quad k\neq r_1\neq r_2$	$\left(\frac{c}{(b-k^2)^2+(2ak)^2}\right)\left((b-k^2)\cos kx+(2ak)\sin kx\right)$
$y''+2ay'+by=c\cos kx \quad k=r_1=r_2$	$-\frac{c}{2k^2}\sin kx$
$y''+k^2y=c\cos kx$	$\frac{c}{2k}x\sin kx$
$y''+2ay'+by=c\sin kx \quad k\neq r_1\neq r_2$	$\left(\frac{c}{(b-k^2)^2+(2ak)^2}\right)\left((-2ak)\cos kx+(b-k^2)\sin kx\right)$
$y''+2ay'+by=c\sin kx \quad k=r_1=r_2$	$\frac{c}{2k^2}\cos kx$
$y''+k^2y=c\sin kx$	$-\frac{c}{2k}x\cos kx$
$y''+2ay'+by=cx^2+dx+e$	$\frac{c}{b}x^2+\frac{bd-4ac}{b^2}x+\frac{8a^2c-2abd+b^2e-2bc}{b^3}$
$y''+2ay'=cx^2+dx+e$	$\frac{c}{6a}x^3+\frac{ad-c}{4a^2}x^2+\frac{2a^2e-ad+c}{4a^3}x$
$y''=cx^2+dx+e$	$\frac{c}{12}x^4+\frac{d}{6}x^3+\frac{e}{2}x^2$

5- Higher Order Differential Equations

The form of a linear nth-order ***homogeneous*** differential equation:

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+a_{n-2}(x)\frac{d^{n-2}y}{dx^{n-2}}+\dots\dots\dots+a_1(x)\frac{dy}{dx}+a_o(x)y=0$$

The form of a linear nth-order ***nonhomogeneous*** differential equation:

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+a_{n-2}(x)\frac{d^{n-2}y}{dx^{n-2}}+\dots\dots\dots+a_1(x)\frac{dy}{dx}+a_o(x)y=f(x)$$

If the functions $a_i(x)$, i = 0, 1, 2, . . ., n are ***constants***, the equation is said to have ***constant coefficients***.

5.1- Homogeneous Higher Order Constant Coefficient Equations

The form of a linear nth-order ***homogeneous*** differential equation:

$$a_n\frac{d^n y}{dx^n}+a_{n-1}\frac{d^{n-1}y}{dx^{n-1}}+a_{n-2}\frac{d^{n-2}y}{dx^{n-2}}+\dots\dots\dots+a_1\frac{dy}{dx}+a_o y=0$$

The solution of the homogeneous equation is

$$y_h=y_1+y_2+y_3+\dots..+y_n$$

Type of Roots	Roots of *r*	Solution
Real	$r=r_1$ $r=r_2$ $r=r_3$ $\vdots$ $r=r_k$	$y_1=C_1\,e^{r_1x}$ $y_2=C_2\,e^{r_2x}$ $y_3=C_3\,e^{r_3x}$ $\vdots$ $y_k=C_k\,e^{r_kx}$
Real and equal	$r=r_1=r_2=r_3\dots.=r_k$	$y_1=C_1\,e^{rx}$ $y_2=C_2\,xe^{rx}$ $y_3=C_3\,x^2e^{rx}$ $\vdots$ $y_k=C_k\,x^{k-1}e^{rx}$
Complex conjugate	$r_1=\alpha_1\pm\beta_1 i$ $r_2=\alpha_2\pm\beta_2 i$ $r_3=\alpha_3\pm\beta_3 i$ $\vdots$ $r_k=\alpha_k\pm\beta_k i$	$y_1+y_2=e^{\alpha_1x}(C_1\cos\beta_1x+C_2\sin\beta_1x)$ $y_3+y_4=e^{\alpha_2x}(C_3\cos\beta_2x+C_4\sin\beta_2x)$ $y_5+y_6=e^{\alpha_3x}(C_5\cos\beta_3x+C_6\sin\beta_3x)$ $\vdots$ $y_{2k-1}+y_{2k}=e^{\alpha_kx}(C_{2k-1}\cos\beta_kx+C_{2k}\sin\beta_kx)$

Complex conjugate and equal	$r = r_1 = r_2 = r_3 \; ... \; = r_k = \alpha \pm \beta i$	$y_1 + y_2 = e^{\alpha x}\left(C_1 \cos \beta x + C_2 \sin \beta x\right)$ $y_3 + y_4 = xe^{\alpha x}\left(C_3 \cos \beta x + C_4 \sin \beta x\right)$ $y_5 + y_6 = x^2 e^{\alpha x}\left(C_5 \cos \beta x + C_6 \sin \beta x\right)$ $\vdots$ $y_{2k-1} + y_{2k} = x^{k-1} e^{\alpha x}\left(C_{2k-1} \cos \beta x + C_{2k} \sin \beta x\right)$

Example 8 : Solve differential equation

(a) $y''' - 2y'' - 5y' + 6y = 0$

(b) $y''' + 2y'' + 4y' = 0$

(c) $y^{(4)} + 2y'' + y = 0$

(d) $6y^{(4)} + 7y''' - 13y'' - 4y' + 4y = 0$

(e) $2y^{(5)} - 7y^{(4)} + 12y''' + 8y'' = 0$

Solution : (a) $y''' - 2y'' - 5y' + 6y = 0$

$y''' - 2y'' - 5y' + 6y = 0 \qquad D^3 y - 2D^2 y - 5Dy + 6y = 0 \qquad \left(D^3 - 2D^2 - 5D + 6\right)y = 0$

$r^3 - 2r^2 - 5r + 6 = 0$

$a = \pm 1$

$b = \pm 1, \pm 2, \pm 3, \pm 6$

$\dfrac{b}{a} = \pm 1, \pm 2, \pm 3, \pm 6$

$Try\ (+1)$

$(1)^3 - 2(1)^2 - 5(1) + 6 = 0 \quad O.K$

Synthetic Division

1	1	−2	−5	6
	↓	1	−1	−6
	1	−1	−6	0

$(r-1)\left(r^2 - r - 6\right) = 0 \qquad (r-1)(r-3)(r+2) = 0$

$r_1 = 1 \qquad y_1 = C_1 e^{x}$

$r_2 = 3 \qquad y_2 = C_2 e^{3x}$

$r_3 = -2 \qquad y_3 = C_3 e^{-2x}$

$\therefore \; y = C_1 e^{x} + C_2 e^{3x} + C_3 e^{-2x}$

Solution : (b) $y''' + 2y'' + 4y' = 0$

Solution : (c) $y^{(4)} + 2y'' + y = 0$

Solution : (d) $6y^{(4)} + 7y''' - 13y'' - 4y' + 4y = 0$

Solution : (f) $2y^{(5)} - 7y^{(4)} + 12y''' + 8y'' = 0$

H.W 7 : Solve differential equation

(a) $y''' + 4y'' + 4y' = 0$

(b) $y^{(4)} + y'' - 2y = 0$

(c) $y^{(6)} + 12y^{(4)} + 48y + 64y = 0$

(d) $y''' + 3y'' + 3y' + y = 0$

(e) $y''' + 4y'' - 7y' - 10y = 0$

(f) $y^{(5)} + 5y^{(4)} - 2y''' - 10y'' + y' + 5y = 0$

(g) $y^{(6)}+y^{(5)}-3y^{(4)}-5y^{(3)}-2y^{(2)}=0$

5.2- Nonhomogeneous Higher Order Constant Coefficient Equations

The form of a linear nth-order ***Nonhomogeneous*** differential equation:

$$a_n\frac{d^ny}{dx^n}+a_{n-1}\frac{d^{n-1}y}{dx^{n-1}}+a_{n-2}\frac{d^{n-2}y}{dx^{n-2}}+.........+a_1\frac{dy}{dx}+a_oy=f(x)$$

The solution of the Nonhomogeneous equation is

$$y=y_h+y_p$$

5.2.1 Undetermined Coefficients Method

$F(x)$	Condition		y_p
k			A
x^m	$r\neq 0$	r are all roots	$A_0+A_1x+A_2x^2+.....+A_{m-1}x^{m-1}+A_mx^m$
	$r=0$	r is one root $[repet(s)ones]$	$x^s\left(A_0+A_1x+A_2x^2+.....+A_{m-1}x^{m-1}+A_mx^m\right)$
x^me^{kx}	$k\neq r$	r are all roots	$\left(A_0+A_1x+A_2x^2+.....+A_mx^m\right)e^{kx}$
	$k=r$	r is one root (e^{rx})	$x^s\left(A_0+A_1x+A_2x^2+.....+A_mx^m\right)e^{kx}$
$x^m\sin kx$ $x^m\cos kx$	$k\neq\beta$	β of all roots	$\left(A_0+A_1x+A_2x^2+.....+A_mx^m\right)\cos kx$ $+\left(B_0+B_1x+B_2x^2+.....+B_mx^m\right)\sin kx$
	$k=\beta$	$(\cos\beta x,\ \sin\beta x)$	$x^s\left(A_0+A_1x+A_2x^2+.....+A_mx^m\right)\cos kx$ $+x^s\left(B_0+B_1x+B_2x^2+.....+B_mx^m\right)\sin kx$
$e^{px}\sin kx$ $e^{px}\cos kx$	$p\neq\alpha\ \&\ k\neq\beta$	$\alpha\ \&\ \beta$ of all roots	$e^{px}\left(A_0\cos kx+B_0\sin kx\right)$
	$p=\alpha\ \&\ k=\beta$	$\alpha\ \&\ \beta$ of one roots $(e^{\alpha x}\cos\beta x, e^{\alpha x}\sin\beta x)$	$x^se^{px}\left(A_0\cos kx+B_0\sin kx\right)$
$e^{px}x^m\sin kx$ $e^{px}x^m\cos kx$	$p\neq\alpha\ \&\ k\neq\beta$	$\alpha\ \&\ \beta$ of all roots	$e^{px}\left(A_0+A_1x+A_2x^2+.....+A_mx^m\right)\cos kx$ $+e^{px}\left(B_0+B_1x+B_2x^2+.....+B_mx^m\right)\sin kx$
	$p=\alpha\ \&\ k=\beta$	$\alpha\ \&\ \beta$ of one roots $(e^{\alpha x}\cos\beta x, e^{\alpha x}\sin\beta x)$	$x^se^{px}\left(A_0+A_1x+A_2x^2+.....+A_mx^m\right)\cos kx$ $+x^se^{px}\left(B_0+B_1x+B_2x^2+.....+B_mx^m\right)\sin kx$

Example 9 : Solve differential equation

(a) $y''' - 5y'' + 6y' = x^2 + \sin x$

(b) $y''' + y'' = e^x \cos x$

(c) $y^{(4)} - y'' = 4x + 2xe^{-x}$

Solution : (a) $y''' - 5y'' + 6y' = x^2 + \sin x$

Solution : (b) $y''' + y'' = e^x \cos x$

Solution : (c) $y^{(4)} - y'' = 4x + 2xe^{-x}$

H.W 8 : Solve differential equation

(a) $y''' + y'' - 2y = 3 + 2\cos x$ (b) $y''' + y'' - y' - y = 2 + e^{-x}$

(c) $y^{(4)} + y''' = 1 - x^2 e^{-x}$ (d) $y''' - 6y'' = 3 - \cos x$

(e) $y''' - 2y'' - 4y' + 8y = 6xe^{2x}$ (f) $y''' - y'' - 4y' + 4y = 5 - e^{x} + e^{2x}$

(g) $y^{(4)} + 2y'' + y = (x-1)^2$ (g) $y''' - 3y'' + 3y' - y = x - 4e^{x}$

5.2.2 Variation of Parameters Method

$$y_h = c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4 + \ldots + c_n y_n$$

$$y_p = u_1 y_1 + u_2 y_2 + u_3 y_3 + u_4 y_4 + \ldots + u_n y_n$$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 & \cdots & y_n \\ y_1' & y_2' & y_3' & y_4' & \cdots & y_n' \\ y_1'' & y_2'' & y_3'' & y_4'' & \cdots & y_n'' \\ y_1''' & y_2''' & y_3''' & y_4''' & \cdots & y_n''' \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & y_3^{(n-2)} & y_4^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & y_4^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 & y_4 & \cdots & y_n \\ 0 & y_2' & y_3' & y_4' & \cdots & y_n' \\ 0 & y_2'' & y_3'' & y_4'' & \cdots & y_n'' \\ 0 & y_2''' & y_3''' & y_4''' & \cdots & y_n''' \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & y_3^{(n-2)} & y_4^{(n-2)} & \cdots & y_n^{(n-2)} \\ f(x) & y_2^{(n-1)} & y_3^{(n-1)} & y_4^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = (-1)^{n+1} f(x) \begin{vmatrix} y_2 & y_3 & y_4 & \cdots & y_n \\ y_2' & y_3' & y_4' & \cdots & y_n' \\ y_2'' & y_3'' & y_4'' & \cdots & y_n'' \\ y_2''' & y_3''' & y_4''' & \cdots & y_n''' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_2^{(n-2)} & y_3^{(n-1)} & y_4^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

$$W_2 = \begin{vmatrix} y_1 & 0 & y_3 & y_4 & \cdots & y_n \\ y_1' & 0 & y_3' & y_4' & \cdots & y_n' \\ y_1'' & 0 & y_3'' & y_4'' & \cdots & y_n'' \\ y_1''' & 0 & y_3''' & y_4''' & \cdots & y_n''' \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & 0 & y_3^{(n-2)} & y_4^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & f(x) & y_3^{(n-1)} & y_4^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = (-1)^{n+2} f(x) \begin{vmatrix} y_1 & y_3 & y_4 & \cdots & y_n \\ y_1' & y_3' & y_4' & \cdots & y_n' \\ y_1'' & y_3'' & y_4'' & \cdots & y_n'' \\ y_1''' & y_3''' & y_4''' & \cdots & y_n''' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_3^{(n-2)} & y_4^{(n-2)} & \cdots & y_n^{(n-2)} \end{vmatrix}$$

$$W_i = \begin{vmatrix} y_1 & y_2 & y_3 & 0 & \cdots & y_n \\ y_1' & y_2' & y_3' & 0 & \cdots & y_n' \\ y_1'' & y_2'' & y_3'' & 0 & \cdots & y_n'' \\ y_1''' & y_2''' & y_3''' & 0 & \cdots & y_n''' \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & y_3^{(n-2)} & 0 & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & f(x) & \cdots & y_n^{(n-1)} \end{vmatrix} = (-1)^{n+i} f(x) \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ y_1'' & y_2'' & y_3'' & \cdots & y_n'' \\ y_1''' & y_2''' & y_3''' & \cdots & y_n''' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & y_3^{(n-2)} & \cdots & y_n^{(n-2)} \end{vmatrix}$$

$$u_1 = \int \frac{W_1}{W}\,dx \quad , \; u_2 = \int \frac{W_2}{W}\,dx \qquad .... \quad u_i = \int \frac{W_i}{W}\,dx \quad .... \quad u_n = \int \frac{W_n}{W}\,dx$$

$$\left[y_p = u_1 y_1 + u_2 y_2 + u_3 y_3 + u_4 y_4 + ... + u_n y_n \right]$$

$$\left[y = y_h + y_p \right]$$

Example 10 : Solve differential equation

(a) $y^{(4)} + 9y'' = \sec^2 3x$

(b) $y''' + y'' = e^x \cos x$

(c) $y^{(4)} - y'' = 4x + 2xe^{-x}$

Solution : (a) $y^{(4)} + 9y'' = \sec^2 3x$

Solution : (b) $y''' + y'' = e^x \cos x$

Solution : (c) $y^{(4)} - y'' = 4x + 2xe^{-x}$

H.W 9 : Solve differential equation

(a) $y^{(4)} - y = 5(x + \cos x)$ (b) $y''' + y'' = e^{x}$

(c) $y^{(4)} - 16y = 1$ (d) $y^{(5)} - y^{(4)} = 1$

(e) $y^{(4)} + 9y'' = 1$ (f) $y''' + 10y'' + 34y' + 40y = xe^{-4x} + 2e^{-3x}\cos x$

(g) $y''' + 6y'' + 11y' + 6y = 2e^{-3x} - xe^{x}$ (h) $y''' - 3y'' + 3y' - y = x - 4e^{x}$

(i) $y^{(4)}+4y''=\sec^2 2x$

(j) $y^{(4)}-10y'''+38y''-64y'+40y=153e^{-x}$

(k) $y^{(4)}+4y''=\tan^2 2x$

(l) $y'''+9y'=\sec^2 2x$

(m) $y'''+6y''-14y'-104y=-111e^{x}$

5.2.3 The Differential Operator (D) Method

$F(x)$	*Differential Operator* D
x^m	D^{m+1}
$x^m e^{kx}$	$(D-k)^{m+1}$
$x^m e^{kx}\cos(cx)$ $x^m e^{kx}\sin(cx)$	$(D^2-2kD+k^2+c^2)^{m+1}$

Example 11-a : Solve differential equation

(a) $y'''-4y''+4y'=5x^2-6x+4x^2e^{2x}+3e^{5x}$

(b) $y'''+8y'=e^{x}\cos 2x$

(c) $y'''+2y''-13y'+10y=xe^{-x}$

Solution : (a) $y'''-4y''+4y'=5x^2-6x+4x^2e^{2x}+3e^{5x}$

Solution : (b) $y''' + 8y' = e^x \cos 2x$

Solution : (c) $y''' + 2y'' - 13y' + 10y = xe^{-x}$

$$y''' + 2y'' - 13y' + 10y = xe^{-x}$$
$$m = 1$$
$$k = -1$$
$$(D-k)^{m+1} = (D+1)^2$$
$$(D+1)^2(D^3 + 2D^2 - 13D + 10)y = 0$$

Synthetic Division

1	1	2	−13	10
	↓	1	3	−10
	1	3	−10	0

$$(r+1)^2(r^3 + 2r^2 - 13r + 10) = 0$$
$$(r+1)^2(r-1)(r^2 + 3r - 10) = 0$$
$$(r+1)^2(r-1)(r-2)(r+5) = 0$$
$$r_1 = -1 \quad \Rightarrow \quad y_{p1} = C_1 e^{-x}$$
$$r_2 = -1 \quad \Rightarrow \quad y_{p2} = C_2 x e^{-x}$$
$$r_3 = 1 \quad \Rightarrow \quad y_{p3} = C_3 e^{x}$$
$$r_4 = 2 \quad \Rightarrow \quad y_{p4} = C_4 e^{2x}$$
$$r_5 = -5 \quad \Rightarrow \quad y_{p5} = C_5 e^{-5x}$$
$$y = C_1 e^{-x} + C_2 x e^{-x} + C_3 e^{x} + C_4 e^{2x} + C_5 e^{-5x} \qquad \triangleleft Ans.$$

H.W 10 : Solve differential equation

(a) $y''' + 8y'' = 2 + 9x - 6x^2$ (b) $y''' - y'' + y' - y = xe^{x} - e^{-x} + 7$

(c) $y''' - 3y'' + 3y'' - y = e^{x} - x + 16$ (d) $2y''' - 3y'' - 3y' + 2y = (e^{x} + e^{-x})^2$

(e) $y^{(4)} - 4y'' = 5x^2 - e^{2x}$ (f) $y''' + 10y'' + 25y' = e^{x}$

(g) $y^{(4)} + 8y' = 4$ (h) $y''' + 4y'' + 3y' = x^2 \cos x - 3x$

(i) $y^{(4)} - 8y'' + 16y = \left(x^3 - 2x\right)e^{4x}$ (j) $y^{(4)} - 10y''' + 38y'' - 64y' + 40y = 153e^{-x}$

(m) $y^{(4)} - 2y''' + y'' = e^x + 1$

5.2.4 Inverse Operator (D^{-1}) Method

$$D = \frac{d}{dx} \qquad D^{-1} = \int dx$$

$$D.E = f(x)$$

$$P(D)y_p = f(x)$$

$$y_p = \frac{1}{P(D)} f(x)$$

General Particular solution y_p

$$y_p = e^{r_n x}\int\left(e^{(r_{n-1}-r_n)x}\int\left(e^{(r_{n-2}-r_{n-1})x}\int\left(e^{(r_{n-3}-r_{n-2})x}\int ....e^{(r_1-r_2)x}\int\left(e^{-r_1 x}f(x)\right)dx\right)dx\right)dx\right)dx$$

$f(x)$	Particular solution y_p		
x^m	$\frac{1}{P(D)}x^m = \left(a_1 + a_2 D + a_3 D^2 ..... + a_m D^m\right)x^m$		
e^{kx}	$\frac{1}{P(k)}e^{kx}$	$k \neq r$	
	$\frac{x^s}{P^{(s)}(k)}e^{kx}$	$k = r$ *is one root* $\left[repet(s)ones\right]$	
$\cos(cx)$ $\sin(cx)$	$\frac{1}{P(-c^2)}\cos(cx)$	*or* $\sin(cx)$	$P(-c^2) \neq 0$
	$\frac{x^s}{P^{(s)}(-c^2)}\cos(cx)$	*or* $\sin(cx)$	$P(-c^2) = 0\left[repet(s)ones\right]$
$x^m e^{kx}$	$e^{kx}\frac{1}{P(D+k)}x^m$		
$e^{kx}\cos(cx)$ $e^{kx}\sin(cx)$	$e^{kx}\frac{1}{P(D+k)}\cos(cx)$	*or* $\sin(cx)$	
$e^{kx}g(x)$	$e^{kx}\frac{1}{P(D+k)}g(x)$		

Note :-

$$\cos(cx) = \frac{1}{2}\left(e^{cix} + e^{-cix}\right)$$

$$\sin(cx) = \frac{i}{2}\left(e^{cix} - e^{-cix}\right)$$

Example 11-b : Solve differential equation

(a) $y''' - 4y'' + 4y' = e^{5x} + 3e^{2x}$

(b) $y''' + 9y' = x^4 + 3x^2 - 2$

(c) $y''' + 4y' = \sin 3x + 5\cos 2x$

(d) $y^{(4)} - 8y'' + 16y = e^{4x}\cos 2\sqrt{3}$

Solution : (a) $y''' - 4y'' + 4y' = e^{5x} + 3e^{2x}$

$y''' - 4y'' + 4y' = e^{5x} + 3e^{2x}$

1-find y_h

$y''' - 4y'' + 4y' = 0$

$(D^3 - 4D^2 + 4D)y_h = 0$

$r^3 - 4r^2 + 4r = 0$

$r(r^2 - 4r + 4) = 0$

$r(r-2)^2 = 0$

$r_1 = 0 \qquad r_2 = 2 \qquad r_3 = 2$

$y_h = C_1 + C_2e^{2x} + C_3xe^{2x}$

1-find y_p

$P(D) = D^3 - 4D^2 + 4D$

$f_1(x) = e^{5x} \quad \Rightarrow \quad k = 5 \qquad y_{p1} = \dfrac{1}{P(5)}e^{5x} = \dfrac{1}{5^3 - 4(5)^2 + 4(5)}e^{5x} = \dfrac{1}{45}e^{5x}$

$f_2(x) = 3e^{2x} \quad \Rightarrow \quad k = 2 = r \qquad \therefore s = 2$

$P'(D) = 3D^2 - 8D$

$P''(D) = 6D - 8$

$y_{p2} = \dfrac{x^2}{P''(2)}(3e^{2x}) = \dfrac{3x^2}{6(2)-8}e^{5x} = \dfrac{3}{4}x^2e^{5x}$

$\therefore y_p = y_{p1} + y_{p2} = \dfrac{1}{45}e^{5x} + \dfrac{3}{4}x^2e^{5x}$

$\therefore y = y_h + y_p = C_1 + C_2e^{2x} + C_3xe^{2x} + \dfrac{1}{45}e^{5x} + \dfrac{3}{4}x^2e^{5x} \qquad \triangleleft Ans.$

Solution : (b) $y'''+9y'=x^4+3x^2-2$

$$y'''+9y'=x^4+3x^2-2$$

1-find y_h

$$y'''+9y'=0$$

$$\left(D^3+9D\right)y_h=0$$

$$r^3+9r=0$$

$$r\left(r^2+9\right)=0$$

$$r_1=0 \qquad r_{2,3}=\pm 3i$$

$$y_h=C_1+C_2\cos 3x+C_3\sin 3x$$

1-find y_p

$$P(D)=D^3+9D$$

$$f(x)=x^4+3x^2-2 \quad \Rightarrow \quad m=4$$

$$\begin{array}{r|l} & \frac{1}{9D}-\frac{D}{81}+\frac{D^3}{729} \\ 9D+D^3 & 1 \\ & 1+\frac{D^2}{9} \\ \hline & 0-\frac{D^2}{9} \\ & -\frac{D^2}{9}-\frac{D^4}{81} \\ \hline & 0+\frac{D^4}{81} \\ & \frac{D^4}{81}+\frac{D^6}{729} \\ \hline & -\frac{D^6}{729} \end{array}$$

$$y_p=\left(\frac{1}{9D}-\frac{D}{81}+\frac{D^3}{729}\right)\left(x^4+3x^2-2\right)=\frac{1}{9}D^{-1}\left(x^4+3x^2-2\right)-\frac{1}{81}D\left(x^4+3x^2-2\right)+\frac{1}{729}D^3\left(x^4+3x^2-2\right)$$

$$y_p=\frac{1}{9}\left(\frac{1}{5}x^5+x^3-2x\right)-\frac{1}{81}\left(4x^3+6x\right)+\frac{1}{729}(24)=\frac{1}{45}x^5+\frac{5}{81}x^3-\frac{64}{243}x$$

$$\therefore y=y_h+y_p=C_1+C_2\cos 3x+C_3\sin 3x+\frac{1}{45}x^5+\frac{5}{81}x^3-\frac{64}{243}x \qquad \lhd Ans.$$

Solution : (c) $y''' + 4y' = \sin 3x + 5\cos 2x$

$$y''' + 4y' = \sin 3x + 5\cos 2x$$

1-find y_h

$$y''' + 4y' = 0$$

$$\left(D^3 + 4D\right) y_h = 0$$

$$r^3 + 4r = 0$$

$$r\left(r^2 + 4\right) = 0$$

$$r_1 = 0 \qquad r_{2,3} = \pm 2i$$

$$y_h = C_1 + C_2 \cos 2x + C_3 \sin 2x$$

1-find y_p

$$P(D) = D^3 + 4D = D\left(D^2 + 4\right)$$

$$f_1(x) = \sin 3x \quad \Rightarrow \quad D^2 = -9 \qquad y_{p1} = \frac{1}{D\left(D^2+4\right)} \sin 3x = \frac{1}{D(-9+4)} \sin 3x$$

$$y_{p1} = \frac{-1}{5} D^{-1}(\sin 3x) = \frac{-1}{5}(-3\cos 3x) = \frac{3}{5}\cos 3x$$

$$f_2(x) = 5\cos 2x \quad \Rightarrow \quad D^2 = -4 \qquad P\left(D^2 = -4\right) = 0 \qquad s = 1$$

$$y_{p2} = \frac{x}{P'(D)}(5\cos 2x) = \frac{x}{3D^2 + 4}(5\cos 2x) = \frac{x}{3(-4)+4}(5\cos 2x) = -\frac{5}{8} x \cos 2x$$

$$\therefore y_p = y_{p1} + y_{p2} = -\frac{3}{5}\cos 3x - \frac{5}{8} x \cos 2x$$

$$\therefore y = y_h + y_p = C_1 + C_2 \cos 2x + C_3 \sin 2x + \frac{3}{5}\cos 3x - \frac{5}{8} x \cos 2x \qquad \triangleleft Ans.$$

Solution : (d) $y^{(4)}-8y''+16y=e^{4x}\cos 2\sqrt{3}$

$$y^{(4)}-8y''+16y=e^{4x}\cos 2\sqrt{3}$$

1-find y_h

$$y^{(4)}-8y''+16y=0$$

$$\left(D^4-8D^2+16\right)y_h=0$$

$$r^4-8r^2+16=0$$

$$\left(r^2-4\right)^2=0$$

$$r_1=r_2=2 \qquad r_3=r_4=-2$$

$$y_h=C_1e^{2x}+C_2xe^{2x}+C_3e^{-2x}+C_4xe^{-2x}$$

1-find y_p

$$P(D)=D^4-8D^2+16=\left(D^2-4\right)^2$$

$$f(x)=e^{4x}\cos 2\sqrt{3} \qquad \Rightarrow \qquad k=4$$

$$P(D+k)=P(D+4)=\left((D+4)^2-4\right)^2=\left(D^2+8D+16-4\right)^2=\left(D^2+8D+12\right)^2$$

$$y_p=e^{4x}\frac{1}{\left(D^2+8D+12\right)^2}\cos 2\sqrt{3}=e^{4x}\frac{1}{\left(-\left(2\sqrt{3}\right)^2+8D+12\right)^2}\cos 2\sqrt{3}=e^{4x}\frac{1}{\left(-12^2+8D+12\right)^2}\cos 2\sqrt{3}$$

$$y_p=e^{4x}\frac{1}{64D^2}\left(\cos 2\sqrt{3}\right)=\frac{1}{64}e^{4x}D^{-2}\left(\cos 2\sqrt{3}\right)=\frac{1}{64}e^{4x}\left(\frac{-1}{12}\cos 2\sqrt{3}\right)=-\frac{1}{768}e^{4x}\cos 2\sqrt{3}$$

$$\therefore y=y_h+y_p=C_1e^{2x}+C_2xe^{2x}+C_3e^{-2x}+C_4xe^{-2x}-\frac{1}{768}e^{4x}\cos 2\sqrt{3} \qquad \triangleleft Ans.$$

5.3- CAUCHY-EULER EQUATION

Cauchy-Euler Equation is a linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_{n-2} x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + ... + a_1 x \frac{dy}{dx} + a_o y = f(x)$$

Let $x = e^z \Leftrightarrow z = \ln x$ and new differential operator $D = \frac{d}{dz}$ and $y = y(z)$

$$a_1 x \frac{dy}{dx} = a_1 D y$$

$$a_2 x^2 \frac{d^2 y}{dx^2} = a_2 D(D-1) y$$

$$a_3 x^3 \frac{d^3 y}{dx^3} = a_3 D(D-1)(D-2) y$$

$$\vdots$$

$$a_k x^k \frac{d^k y}{dx^k} = a_k D(D-1)(D-2)(D-3)\ldots \ldots + (D-k+1) y \qquad where \quad k = n - Order\, of \;\, D.E$$

Example 12 : Solve differential equation

(a) $x^2 y'' - 2xy' - 4y = 0$

(b) $x^3 y''' + 5x^2 y'' + 7xy' + 8y = 0$

(c) $x^4 y^{(4)} + 6x^3 y''' + 9x^2 y'' + 3xy' + y = 0$

(d) $x^2 y'' - 3xy' + 3y = 2x^4$

(e) $x^2 y'' - xy' + y = \sqrt{x} + \ln x$

Solution : (a) $x^2 y'' - 2xy' - 4y = 0$

$$x^2 y'' - 2xy' - 4y = 0$$

$$D(D-1)y - 2Dy - 4y = 0$$

$$r(r-1) - 2r - 4 = 0$$

$$r^2 - r - 2r - 4 = 0$$

$$r^2 - 3r - 4 = 0$$

$$(r-4)(r+1) = 0$$

$$r_1 = 4 \qquad\qquad r_2 = -1$$

$$y = c_1 e^{4z} + c_2 e^{-z}$$

$$y(z) = c_1 (e^z)^4 + c_2 (e^z)^{-1}$$

$$\because x = e^z$$

$$y(x) = c_1 x^4 + c_2 x^{-1}$$

Solution : (b) $x^3y'''+5x^2y''+7xy'+8y=0$

$$x^3y'''+5x^2y''+7xy'+8y=0$$
$$D(D-1)(D-2)y+5D(D-1)y+7Dy+8y=0$$
$$r(r-1)(r-2)+5r(r-1)+7r+8=0$$
$$r^3+2r+4r+8=0$$
$$(r+2)(r^2+4)=0$$
$$r_1=-2 \qquad r_{2,3}=\pm 2i$$
$$y(z)=c_1e^{-z}+c_2\cos 2z+c_3\sin 2z$$
$$\because x=e^z \qquad z=\ln x$$
$$y(x)=\frac{c_1}{x}+c_2\cos(2\ln x)+c_3\sin(2\ln x)$$

Solution : (c) $x^4y^{(4)}+6x^3y'''+9x^2y''+3xy'+y=0$

$$x^4y^{(4)}+6x^3y'''+9x^2y''+3xy'+y=0$$
$$D(D-1)(D-2)(D-3)y+6D(D-1)(D-2)y+9D(D-1)+3Dy+y=0$$
$$r(r-1)(r-2)(r-3)+6r(r-1)(r-2)+9r(r-1)+3r+1=0$$
$$r^4+2r^2+1=0$$
$$(r^2+1)^2=0$$
$$r_{1,2}=r_{3,4}=i$$
$$y(z)=c_1\cos z+c_2\sin z+z(c_3\cos z+c_4\sin z)$$
$$\because x=e^z \qquad z=\ln x$$
$$y(z)=c_1\cos(\ln x)+c_2\sin(\ln x)+(\ln x)(c_3\cos(\ln x)+c_4\sin(\ln x))$$

Solution : (d) $x^2y''-3xy'+3y=2x^4e^x$

$$x^2y''-3xy'+3y=2x^4$$

$$Let \qquad x=e^z \qquad\qquad z=\ln x$$

$$D(D-1)y-3Dy+3y=2e^{4z}$$

1–find y_h

$$r(r-1)-3r+3=0$$

$$r^2-4r+3=0$$

$$(r-1)(r-3)=0$$

$$r_1=1 \qquad\qquad r_{2,3}=3$$

$$y(z)=c_1e^z+c_2e^{3z}$$

1–find y_p

$$P(D)=D^2-4D+3 \qquad\qquad k=4$$

$$y_p=\frac{1}{P(k)}e^{kz}=\frac{1}{P(4)}\left(2e^{4z}\right)=\frac{1}{4^2-4\times4+3}\left(2e^{4z}\right)=\frac{2}{3}e^{4z}$$

$$y(z)=y_h+y_p=c_1e^z+c_2e^{3z}+\frac{2}{3}e^{4z}$$

$$\because x=e^z$$

$$y(x)=c_1x+c_2x^3+\frac{2}{3}x^4$$

Solution : (e) $x^2y''-xy'+y=\sqrt{x}+\ln x$

$$x^2y''-xy'+y=\sqrt{x}+\ln x$$

$$Let \quad x=e^z \qquad z=\ln x$$

$$D(D-1)-Dy+y=e^{\frac{1}{2}z}+z$$

1–find y_h

$$D(D-1)-Dy+y=0$$

$$r(r-1)-r+1=0$$

$$(r-1)^2=0$$

$$r_{1,2}=1$$

$$y(z)=c_1e^z+c_2ze^z$$

1–find y_p

$$P(D)=(D-1)^2$$

$$k=\frac{1}{2} \qquad y_{p1}=\frac{1}{P(k)}e^{\frac{1}{2}z}=\frac{1}{P\left(\frac{1}{2}\right)}\left(e^{\frac{1}{2}z}\right)=\frac{1}{\left(\frac{1}{2}-1\right)^2}\left(e^{\frac{1}{2}z}\right)=4e^{\frac{1}{2}z}$$

$$y_{p2}=\frac{1}{(D-1)^2}z=(-1-D)^2z=(1+2D+D^2)z=z+2$$

$$y_p=4e^{\frac{1}{2}z}+z+2$$

$$y(z)=y_h+y_p=c_1e^z+c_2ze^z+4e^{\frac{1}{2}z}+z+2$$

$$\because x=e^z \qquad z=\ln x$$

$$y(z)=c_1x+c_2x\ln x+4\sqrt{x}+\ln x+2$$

H.W 11 : Solve differential equation

(a) $x^2y''-xy'+y=\ln x$

(b) $x^3y'''-6y=0$

(c) $xy^{(4)}+6y'''=0$

(d) $x^3y'''-3x^2y''+6xy'-6y=3+\ln x^3$

(e) $x^2y''-4xy'+6y=\ln x^2$

(f) $x^2y''-(x^2+2x)y'+(x+2)y=x^3$

(e) $x^2y''+xy'-y=\dfrac{1}{1+x}$

(f) $(x-2)^2y''+(x+2)y'+y=0$

(e) $(2x+1)^2y''-2(2x+1)y'+y=2x$

6- Reduction of Order

The general solution of a homogeneous linear second-order differential equation

$$g_2(x)\frac{d^2y}{dx^2}+g_1(x)\frac{dy}{dx}+g_o(x)y=0$$

$$y=c_1y_1+c_2y_2$$

y_1 is given

$$\therefore\ y_2=uy_1$$

$$y_2=uy_1$$

the standard form

$$y''+P(x)y'+Q(x)y=0$$

$$\therefore\ y_2=y_1\int\frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$$

Example 13 : find y_2 for differential equation if y_1 is given

(a) $y''-y=0$ $\qquad y_1=e^x$

(b) $x^2y''-3xy'+4y=0$ $\qquad y_1=x^2$

(c) $xy''+y'=0$ $\qquad y_1=\ln x$

Solution : (a) $y''-y=0$ $\qquad y_1=e^x$

$$P(x)=0$$

$$y_2=y_1\int\frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx=e^x\int\frac{e^{-\int 0\,dx}}{e^{2x}}\,dx=e^x\int\frac{e^{-\int 0\,dx}}{e^{2x}}\,dx=e^x\int\frac{e^0}{e^{2x}}\,dx$$

$$=e^x\int e^{-2x}\,dx=-\frac{1}{2}e^{-x}$$

$$y=c_1y_1+c_2y_2=y=c_1e^x-\frac{c_2}{2}e^{-x}$$

Solution : (b) $x^2y''-3xy'+4y=0$ $\qquad y_1=x^2$

$$\left(x^2y''-3xy'+4y=0\right)\qquad \div x^2$$

$$y''-\frac{3}{x}y'+\frac{4}{x^2}y=0\qquad P(x)=-\frac{3}{x}$$

$$y_2=y_1\int\frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx=x^2\int\frac{e^{-\int -\frac{3}{x}\,dx}}{\left(x^2\right)^2}\,dx=x^2\int\frac{e^{3\ln x}}{x^4}\,dx=x^2\int\frac{e^{\ln x^3}}{x^4}\,dx$$

$$=x^2\int\frac{1}{x}\,dx=x^2\ln x$$

$$y=c_1y_1+c_2y_2=y=c_1x^2+c_2x^2\ln x$$

Solution : (c) $xy''+y'=0$ $\quad y_1=\ln x$

$$xy''+y'=0 \qquad y_1=\ln x$$

$$(xy''+y'=0)\ \div x$$

$$y''+\frac{1}{x}y'=0$$

$$P(x)=\frac{1}{x}$$

$$y_2=y_1\int\frac{e^{-\int P(x)\,dx}}{y_1^2}dx=\ln x\int\frac{e^{-\int\frac{1}{x}dx}}{(\ln x)^2}dx=\ln x\int\frac{e^{-\ln x}}{(\ln x)^2}dx=x^2\int\frac{e^{\ln x^{-1}}}{x^4}dx$$

$$=\ln x\int\frac{1}{x(\ln x)^2}dx=\ln x\left(\frac{-1}{\ln x}\right)=-1$$

$$y=c_1y_1+c_2y_2=y=c_1\ln x-c_2$$

H.W 12 : find y_2 for differential equation if y_1 is given

(a) $y''-4y'+4y=0$ $\quad y_1=e^{2x}$

(b) $y''-y=0$ $\quad y_1=\cosh x$

(c) $4x^2y''+y=0$ $\quad y_1=\sqrt{x}\ln x$

(d) $y''+9y=0$ $\quad y_1=\cos 3x$

(e) $x^2y''-3xy'+5y=0$ $\quad y_1=x^2\cos(\ln x)$

(f) $(1-2x-x^2)y''+2(1+x)y'-2y=0$ $\quad y_1=1+x$

(g) $(1-x^2)y''+2xy'-2xy=0$ $\quad y_1=1+x$

7- System Linear Differential Equations:

$$x = g(t) \qquad y = f(t) \qquad D = \frac{d}{dt}$$

$$aDx + bDy = F_1(t)$$
$$cDx + aDy = F_2(t)$$

$$\begin{vmatrix} aD & bD \\ cD & dD \end{vmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix} \qquad x = \frac{\begin{vmatrix} F_1(t) & bD \\ F_2(t) & dD \end{vmatrix}}{\begin{vmatrix} aD & bD \\ cD & dD \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} aD & F_1(t) \\ cD & F_2(t) \end{vmatrix}}{\begin{vmatrix} aD & bD \\ cD & dD \end{vmatrix}}$$

Example 14 : Solve the system of differential equation

$$2\frac{dx}{dt} + x + \frac{dy}{dt} = t$$
$$3\frac{dx}{dt} + 2\frac{dy}{dt} - y = 0$$

Solution :

$$2Dx + x + Dy = t \qquad (2D+1)x + (D)y = t$$
$$3Dx + 2Dy - y = 0 \qquad (3D)x + (2D-1)y = 0$$

$$W = \begin{vmatrix} 2D+1 & D \\ 3D & 2D-1 \end{vmatrix} = (2D+1)(2D-1) - (3D)(D) = D^2 - 1$$

$$W_x = \begin{vmatrix} t & D \\ 0 & 2D-1 \end{vmatrix} = (2D-1)(t) - (D)(0) = 2D(t) - t = 2 - t$$

$$W_y = \begin{vmatrix} 2D+1 & t \\ 3D & 0 \end{vmatrix} = (2D+1)(0) - (3D)(t) = -3$$

$$x = x_h + x_p$$
$$y = y_h + y_p$$

Homogeneous solution

$$Wx_h = (D^2 - 1)x_h = 0$$
$$Wy_h = (D^2 - 1)y_h = 0$$

$$r^2 - 1 = 0 \qquad r_1 = 1 \qquad r_2 = -1$$

$$x_h = C_1e^t + C_2e^{-t} \qquad y_h = C_1e^t + C_2e^{-t}$$

$$x_p = \frac{W_x}{W} = \frac{2-t}{D^2-1} = (-1-D^2)(2-t) = (-1)(2-t) - D^2(2-t) = t - 2$$

$$y_p = \frac{W_y}{W} = \frac{-3}{D^2-1} = (-1-D^2)(-3) = 3$$

$$x = x_h + x_p = C_1e^t + C_2e^{-t} + t - 2 \qquad \triangleleft Ans$$

$$y = y_h + y_p = C_1e^t + C_2e^{-t} + 3 \qquad \triangleleft Ans$$

Example 15 : Solve the system of differential equation

$$2\frac{dx}{dt}-x+\frac{dy}{dt}=e^t$$
$$3\frac{dx}{dt}+2\frac{dy}{dt}+y=t$$

Solution :

$$2Dx-x+Dy=e^t \qquad (2D-1)x+(D)y=e^t$$
$$3Dx+2Dy+y=t \qquad (3D)x+(2D+1)y=t$$
$$W=\begin{vmatrix}2D-1 & D\\ 3D & 2D+1\end{vmatrix}=(2D-1)(2D+1)-(3D)(D)=D^2-1$$
$$W_x=\begin{vmatrix}e^t & D\\ t & 2D+1\end{vmatrix}=(2D+1)(e^t)-(D)(t)=3e^t-1$$
$$W_y=\begin{vmatrix}2D-1 & e^t\\ 3D & t\end{vmatrix}=(2D-1)(t)-(3D)(e^t)=2-t-3e^t$$
$$x=x_h+x_p$$
$$y=y_h+y_p$$

Homogeneous solution

$$Wx_h=(D^2-1)x_h=0$$
$$Wy_h=(D^2-1)y_h=0$$
$$r^2-1=0 \qquad r_1=1 \qquad r_2=-1$$
$$x_h=C_1e^t+C_2e^{-t} \qquad y_h=C_1e^t+C_2e^{-t}$$
$$x_p=\frac{W_x}{W}=\frac{3e^t-1}{D^2-1}=3\left(\frac{e^t}{D^2-1}\right)-\left(\frac{1}{D^2-1}\right)=3\left(\frac{te^t}{2D}\right)-\left(-1-D^2\right)(1)=3\left(\frac{te^t}{2\times 1}\right)-\left(-1-D^2\right)(1)=\frac{3}{2}te^t+1$$
$$y_p=\frac{W_y}{W}=\frac{2-t-3e^t}{D^2-1}=\left(\frac{2-t}{D^2-1}\right)-3\left(\frac{e^t}{D^2-1}\right)=\left(-1-D^2\right)(2-t)-3\left(\frac{te^t}{2D}\right)=t-2-\frac{3}{2}te^t$$
$$x=x_h+x_p=C_1e^t+C_2e^{-t}+\frac{3}{2}te^t+1 \qquad \triangleleft Ans$$
$$y=y_h+y_p=C_1e^t+C_2e^{-t}+t-2-\frac{3}{2}te^t \qquad \triangleleft Ans$$

Example 16 : Solve the system of differential equation

$$\left(2D^2+3D-9\right)x+\left(D^2+7D-14\right)y=4$$
$$(D+1)x+(D+2)y=-8e^{2t}$$

Solution :

$$W=\begin{vmatrix}2D^2+3D-9 & D^2+7D-14\\ D+1 & D+2\end{vmatrix}=\left(2D^2+3D-9\right)(D+2)-(D+1)\left(D^2+7D-14\right)$$

$$W=D^3-D^2+4D-4=(D-1)\left(D^2+4\right)$$

$$W_x=\begin{vmatrix}4 & D^2+7D-14\\ -8e^{2t} & D+2\end{vmatrix}=(D+2)(4)-\left(D^2+7D-14\right)\left(-8e^{2t}\right)=8+32e^{2t}$$

$$W_y=\begin{vmatrix}2D^2+3D-9 & 4\\ D+1 & -8e^{2t}\end{vmatrix}=\left(2D^2+3D-9\right)\left(-8e^{2t}\right)-(D+1)(4)=-4-40e^{2t}$$

$$x=x_h+x_p$$

$$y=y_h+y_p$$

Homogeneous solution

$$Wx_h=\left(D^3-D^2+4D-4\right)x_h=0$$

$$Wy_h=\left(D^3-D^2+4D-4\right)y_h=0$$

$$(r-1)\left(r^2+4\right)=0 \qquad r_1=1 \qquad r_{2,3}=\pm 2i$$

$$x_h=C_1e^t+C_2\cos 2t+C_3\sin 2t$$

$$y_h=C_1e^t+C_2\cos 2t+C_3\sin 2t$$

$$x_p=\frac{8+32e^{2t}}{D^3-D^2+4D-4}=\left(\frac{1}{D^3-D^2+4D-4}\right)(8)+32\left(\frac{e^{2t}}{D^3-D^2+4D-4}\right)$$

$$=\left(-\frac{1}{4}-\frac{D}{4}\right)(8)+32\left(\frac{e^{2t}}{2^3-2^2+4\times 2-4}\right)=-2+4e^{2t}$$

$$y_p=\frac{-4-40e^{2t}}{D^3-D^2+4D-4}=\left(\frac{1}{D^3-D^2+4D-4}\right)(-4)-40\left(\frac{e^{2t}}{D^3-D^2+4D-4}\right)$$

$$=\left(-\frac{1}{4}-\frac{D}{4}\right)(-4)-40\left(\frac{e^{2t}}{2^3-2^2+4\times 2-4}\right)=1-5e^{2t}$$

$$x=x_h+x_p=C_1e^t+C_2\cos 2t+C_3\sin 2t-2+4e^{2t} \qquad \triangleleft Ans$$

$$y=y_h+y_p=C_1e^t+C_2\cos 2t+C_3\sin 2t+1-5e^{2t} \qquad \triangleleft Ans$$

Example 17 : Solve the system of differential equation

$$(D+1)x+(D+2)y+(D+3)z=-e^{-t}$$
$$(D+2)x+(D+3)y+(2D+3)z=e^{-t}$$
$$(4D+6)x+(5D+4)y+(20D-12)z=7e^{-t}$$

Solution :

$$(D+1)x+(D+2)y+(D+3)z=-e^{-t}$$
$$(D+2)x+(D+3)y+(2D+3)z=e^{-t}$$
$$(4D+6)x+(5D+4)y+(20D-12)z=7e^{-t}$$

$$W=\begin{vmatrix} D+1 & D+2 & D+3 \\ D+2 & D+3 & 2D+3 \\ 4D+6 & 5D+4 & 20D-12 \end{vmatrix} = \begin{vmatrix} D+1 & D+2 & D+3 \\ D+2 & D+3 & 2D+3 \\ 4D+6 & 5D+4 & 20D-12 \end{vmatrix}\begin{matrix} D+1 & D+2 \\ D+2 & D+3 \\ 4D+6 & 5D+4 \end{matrix}$$

$$W=-D^3+6D^2-11D+6$$

$$W_x=\begin{vmatrix} -e^{-t} & D+2 & D+3 \\ e^{-t} & D+3 & 2D+3 \\ 7e^{-t} & 5D+4 & 20D-12 \end{vmatrix}=\left(-18D^2-27D+63\right)\left(e^{-t}\right)=72e^{-t}$$

$$W_y=\begin{vmatrix} D+1 & -e^{-t} & D+3 \\ D+2 & e^{-t} & 2D+3 \\ 4D+6 & 7e^{-t} & 20D-12 \end{vmatrix}=\left(21D^2-6D-51\right)\left(e^{-t}\right)=-24e^{-t}$$

$$W_z=\begin{vmatrix} D+1 & D+2 & -e^{-t} \\ D+2 & D+3 & e^{-t} \\ 4D+6 & 5D+4 & 7e^{-t} \end{vmatrix}=\left(-2D^2+9D+11\right)\left(e^{-t}\right)=0$$

$$x=x_h+x_p$$
$$y=y_h+y_p$$

Homogeneous solution

$$Wx_h=\left(-D^3+6D^2-11D+6\right)x_h=0$$
$$Wy_h=\left(-D^3+6D^2-11D+6\right)y_h=0$$
$$-(r-1)(r-2)(r-3)=0 \qquad r_1=1 \qquad r_2=2 \qquad r_3=3$$
$$x_h=C_1e^{t}+C_2e^{2t}+C_3e^{3t}$$
$$y_h=C_1e^{t}+C_2e^{2t}+C_3e^{3t}$$

$$x_p = \frac{W_x}{W} = \frac{72e^{-t}}{\left(-D^3+6D^2-11D+6\right)} = 72\left(\frac{e^{-t}}{-\left(-1\right)^3+6\left(-1\right)^2-11\left(-1\right)+6}\right) = 3e^{-t}$$

$$y_p = \frac{W_y}{W} = \frac{-24e^{-t}}{\left(-D^3+6D^2-11D+6\right)} = -24\left(\frac{e^{-t}}{-\left(-1\right)^3+6\left(-1\right)^2-11\left(-1\right)+6}\right) = -e^{-t}$$

$$y_z = \frac{W_z}{W} = \frac{0}{\left(-D^3+6D^2-11D+6\right)} = 0$$

$$x = x_h + x_p = C_1e^t + C_2e^{2t} + C_3e^{3t} + 3e^{-t} \qquad \triangleleft Ans$$

$$y = y_h + y_p = C_1e^t + C_2e^{2t} + C_3e^{3t} - e^{-t} \qquad \triangleleft Ans$$

$$z = z_h + z_p = C_1e^t + C_2e^{2t} + C_3e^{3t} \qquad \triangleleft Ans$$

H.W 13 : Solve the system of differential equation

(a) $(D^2+5)x - 2y = 0$
$-2x + (D^2+2)y = 0$

(b) $(D+1)x + (D-1)y = 2$
$3x + (D+2)y = -1$

(c) $D^2x - Dy = t$
$(D+3)x + (D+3)y = 2$

(d) $(2D^2 - D - 1)x - (2D+1)y = 1$
$(D-1)x + Dy = -1$

(e) $D^2x - 2(D^2+D)y = \sin t$
$x + Dy = 0$

(f) $(D+2)x + (D+1)y = \sin 2t$
$5x + (D+3)y = \cos 2t$

(g) $Dx + z = e^t$
$(D-1)x + Dy + Dz = 0$
$x + 2y + Dz = e^t$

8- Determination of D.E. if its solution is known

Example 13 : Find D.E if solution is given

(a) $y = C_1 \cos x + C_2 \sin x - x$

(b) $y = C_1 e^{x} + C_2 e^{2x} + 10$

(c) $y = C_1 e^{x} + C_2 x e^{x} + 4x$

(d) $y = e^{2x}\left(C_1 \cos 3x + C_2 \sin 3x\right) - 5x$

Solution : (a) $y = C_1 \cos x + C_2 \sin x - x$

$$y = C_1 \cos x + C_2 \sin x - x \qquad --------[1]$$
$$y' = -C_1 \sin x + C_2 \cos x - 1 \qquad --------[2]$$
$$y'' = -C_1 \cos x - C_2 \sin x \qquad --------[3]$$

$$eq[3] + eq[1] =$$
$$y'' + y = -C_1 \cos x - C_2 \sin x + C_1 \cos x + C_2 \sin x - x$$
$$y'' + y = -x$$

Solution : (b) $y = C_1 e^{x} + C_2 e^{2x} + 10$

$$y = C_1 e^{x} + C_2 e^{2x} + 10 \qquad --------[1]$$
$$y' = C_1 e^{x} + 2C_2 e^{2x} \qquad --------[2]$$
$$y'' = C_1 e^{x} + 4C_2 e^{2x} \qquad --------[3]$$

$$eq[3] - eq[2] =$$
$$y'' - y' = \left(C_1 e^{x} + 4C_2 e^{2x}\right) - \left(C_1 e^{x} + 2C_2 e^{2x}\right)$$
$$y'' - y' = 2C_2 e^{2x}$$
$$C_2 = \frac{y'' - y'}{2e^{2x}} \qquad sub\ in\ eq[2]$$
$$y' = C_1 e^{x} + 2e^{2x}\left(\frac{y'' - y'}{2e^{2x}}\right)$$
$$C_1 = \frac{y'' + 2y'}{e^{x}} \qquad C_1 \,\&\, C_2 \qquad sub\ in\ eq[1]$$
$$y = C_1 e^{x} + C_2 e^{2x} + 10$$
$$y = \left(\frac{y'' + 2y'}{e^{x}}\right)e^{x} + \left(\frac{y'' - y'}{2e^{2x}}\right)e^{2x} + 10$$
$$y = y'' + 2y' + \frac{y'' - y'}{2} + 10$$
$$y'' - 3y' + 2y = 20 \qquad \lhd Ans.$$

Solution : (c) $y = C_1 e^{x} + C_2 x e^{x} + 4x$

$$y = C_1 e^{x} + C_2 x e^{x} + 4x$$
$$e^{-x} y = C_1 + C_2 x + 4xe^{-x}$$
$$e^{-x} y' - e^{-x} y = C_2 + 4\left(-xe^{-x} + e^{-x}\right)$$
$$\left(e^{-x} y'' - e^{-x} y'\right) - \left(e^{-x} y' - e^{-x} y\right) = 4\left(xe^{-x} - e^{-x} - e^{-x}\right) \qquad \times e^{-x}$$
$$\left(y'' - y'\right) - \left(y' - y\right) = 4\left(x - 2\right)$$
$$y'' - 2y' + y = 4x - 8 \qquad \triangleleft Ans.$$

Solution : (d) $y = e^{2x}\left(C_1 \cos 3x + C_2 \sin 3x\right) - 5x$

$$y = e^{2x}\left(C_1 \cos 3x + C_2 \sin 3x\right) - 5x \qquad ----[1]$$
$$e^{-2x} y = C_1 \cos 3x + C_2 \sin 3x - 5xe^{-2x}$$
$$e^{-2x} y' - 2e^{-2x} y = -3C_1 \sin 3x + 3C_2 \cos 3x - 5\left(-2xe^{-2x} + e^{-2x}\right) \qquad ----[2]$$
$$\left(e^{-2x} y'' - 2e^{-2x} y'\right) - 2\left(e^{-2x} y' - 2e^{-2x} y\right) = -9C_1 \cos 3x - 9C_2 \sin 3x - 5\left(-2\left(-2xe^{-2x} + e^{-2x}\right) - 2e^{-2x}\right)$$
$$e^{-2x}\left(y'' - 4y'' + 4y\right) = 9\left(-C_1 \cos 3x - C_2 \sin 3x\right) + 20e^{-2x}\left(1 - x\right) \qquad \times \frac{e^{2x}}{9}$$
$$\frac{1}{9}\left(y'' - 4y'' + 4y\right) = -e^{2x}\left(C_1 \cos 3x + C_2 \sin 3x\right) + \frac{20}{9}\left(1 - x\right) \qquad ----[3]$$

$$eq[3] + eq[1] =$$
$$\frac{1}{9}\left(y'' - 4y'' + 4y\right) + y = e^{2x}\left(C_1 \cos 3x + C_2 \sin 3x\right) - 5x - e^{2x}\left(C_1 \cos 3x + C_2 \sin 3x\right) + \frac{20}{9}\left(1 - x\right)$$
$$\frac{1}{9}\left(y'' - 4y'' + 4y\right) + y = -5x + \frac{20}{9}\left(1 - x\right) \qquad \times 9$$
$$y'' - 4y'' + 4y + 9y = -45x + 20\left(1 - x\right)$$
$$y'' - 4y'' + 13y = 20 - 65x \qquad \triangleleft Ans.$$

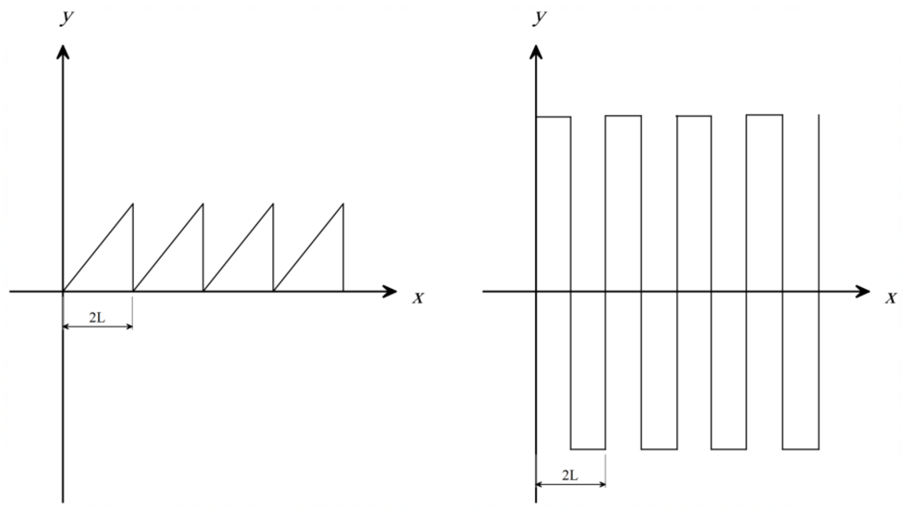
9- FOURIER SERIES

If f(x) is continuous and $-\ell \le x \le \ell$ or $0 \le x \le 2\ell$ then its possible to write :

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}$$

or

$$f(x) = \frac{a_o}{2} + a_1\cos\frac{\pi x}{\ell} + a_2\cos\frac{2\pi x}{\ell} + a_3\cos\frac{3\pi x}{\ell} + a_4\cos\frac{4\pi x}{\ell} + \ldots$$
$$+ b_1\sin\frac{\pi x}{\ell} + b_2\sin\frac{2\pi x}{\ell} + b_3\sin\frac{3\pi x}{\ell} + b_4\sin\frac{4\pi x}{\ell} + \ldots$$



A Fourier series is a representation of a function as a series of constant multiples of sine and/or cosine functions of different frequencies.

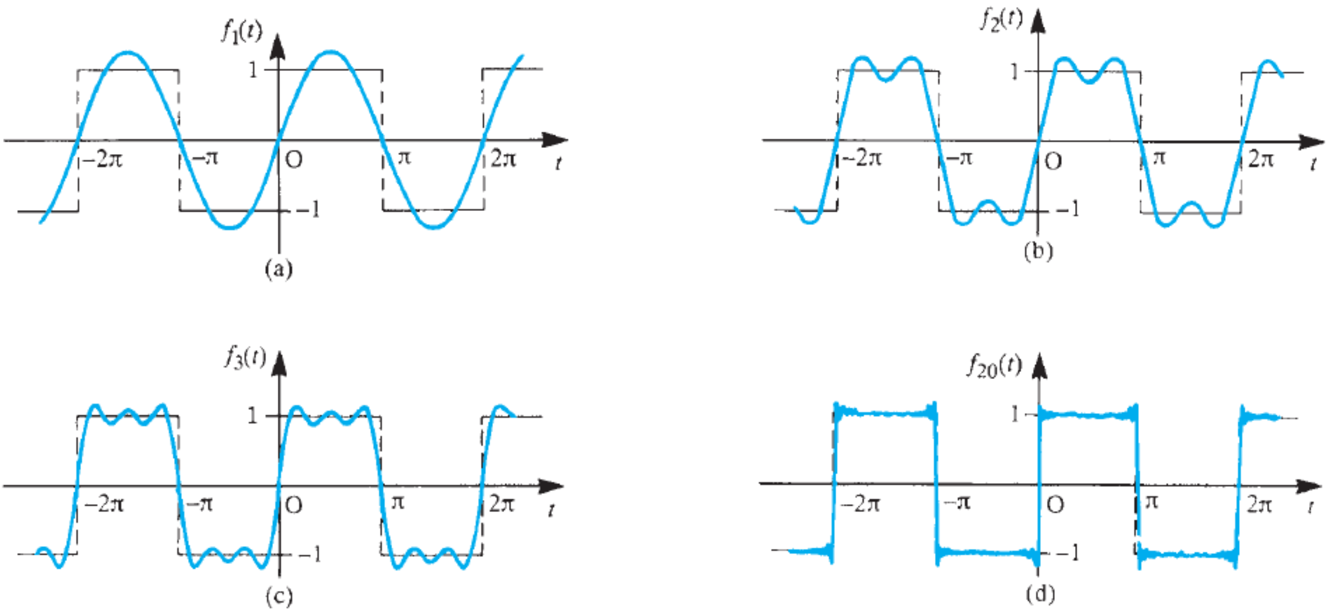


Figure Plots of $f_N(t)$ for a square wave: (a) $N = 1$; (b) 2; (c) 3; (d) 20.

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$
$$\sin(A-B)=\sin A\cos B-\cos A\sin B$$
$$------------------------$$
$$\sin(A+B)+\sin(A-B)=2\sin A\cos B$$
$$\sin A\cos B=\frac{1}{2}\left(\sin(A+B)+\sin(A-B)\right)$$
$$Let \quad u=A+B \qquad v=A-B$$
$$A=\frac{u+v}{2} \qquad B=\frac{u-v}{2}$$
$$\sin u+\sin v=2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$
$$\cos(A-B)=\cos A\cos B+\sin A\sin B$$
$$------------------------$$
$$\cos(A+B)+\cos(A-B)=2\cos A\cos B$$
$$\cos A\cos B=\frac{1}{2}\left(\cos(A+B)+\cos(A-B)\right)$$
$$\cos u+\cos v=2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$
$$\cos u-\cos v=-2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

Integration of Trigonometric Function

$$I=\int_{0}^{2\ell} \quad =\int_{-\ell}^{\ell}$$

(1) $$I=\int_{0}^{2\ell}\sin\frac{n\pi x}{\ell}dx=-\frac{\ell}{n\pi}\cos\frac{n\pi x}{\ell}\Big|_{0}^{2\ell}=-\frac{\ell}{n\pi}\left[\cos n\pi-\cos 0\right]=-\frac{\ell}{n\pi}[1-1]=0$$

(2) $$I=\int_{0}^{2\ell}\cos\frac{n\pi x}{\ell}dx=\frac{\ell}{n\pi}\sin\frac{n\pi x}{\ell}\Big|_{0}^{2\ell}=\frac{\ell}{n\pi}\left[\sin n\pi-\sin 0\right]=\frac{\ell}{n\pi}[0-0]=0$$

(3) $$I=\int_{0}^{2\ell}\sin\frac{n\pi x}{\ell}\sin\frac{m\pi x}{\ell}dx$$

$(i)\quad m\neq n$

$$I=\int_{0}^{2\ell}\sin\frac{n\pi x}{\ell}\sin\frac{m\pi x}{\ell}dx=-\frac{1}{2}\int_{0}^{2\ell}\left[\cos\frac{(n+m)\pi x}{\ell}-\cos\frac{(n-m)\pi x}{\ell}\right]dx$$

$$=-\frac{1}{2}\int_{0}^{2\ell}\cos\frac{(n+m)\pi x}{\ell}dx+\frac{1}{2}\int_{0}^{2\ell}\cos\frac{(n-m)\pi x}{\ell}dx=0+0$$

$(ii)\quad m=n$

$$I=\int_{0}^{2\ell}\sin^2\frac{n\pi x}{\ell}dx=\frac{1}{2}\int_{0}^{2\ell}\left(1-\cos\frac{2n\pi x}{\ell}\right)dx=\frac{1}{2}\int_{0}^{2\ell}dx-\frac{1}{2}\int_{0}^{2\ell}\cos\frac{2n\pi x}{\ell}dx$$

$$=\frac{1}{2}(2\ell)-0=\ell$$

(4) $$I=\int_{0}^{2\ell}\sin\frac{n\pi x}{\ell}\cos\frac{m\pi x}{\ell}dx=0$$

(5) $$I=\int_{0}^{2\ell}\cos\frac{n\pi x}{\ell}\cos\frac{m\pi x}{\ell}dx=\begin{cases}\ell & m=n\\ 0 & m\neq n\end{cases}$$

Determination of Euler Coefficients

$$f(x)=\frac{a_o}{2}+a_1\cos\frac{\pi x}{\ell}+a_2\cos\frac{2\pi x}{\ell}+a_3\cos\frac{3\pi x}{\ell}+a_4\cos\frac{4\pi x}{\ell}+\ldots$$
$$+b_1\sin\frac{\pi x}{\ell}+b_2\sin\frac{2\pi x}{\ell}+b_3\sin\frac{3\pi x}{\ell}+b_4\sin\frac{4\pi x}{\ell}+\ldots$$

(1) $a_o=?$ to find a_o integrate both sides from 0 to 2ℓ

$$I=\int_0^{2\ell} f(x)dx=\frac{a_o}{2}\int_0^{2\ell} dx+a_1\int_0^{2\ell}\cos\frac{\pi x}{\ell}dx+a_2\int_0^{2\ell}\cos\frac{2\pi x}{\ell}dx+\ldots\ldots+a_n\int_0^{2\ell}\cos\frac{n\pi x}{\ell}dx$$
$$+b_1\int_0^{2\ell}\sin\frac{\pi x}{\ell}dx+b_2\int_0^{2\ell}\sin\frac{2\pi x}{\ell}dx+\ldots\ldots+b_n\int_0^{2\ell}\sin\frac{n\pi x}{\ell}dx$$
$$=\frac{a_o}{2}\left(x\Big|_0^{2\ell}\right)dx=a_o\ell$$

$$\therefore \boxed{a_o=\frac{1}{\ell}\int_0^{2\ell} f(x)dx=\frac{1}{\ell}\int_{-\ell}^{\ell} f(x)dx}$$

(2) $a_n=?$ to find a_n multiply eq(1) by $\cos\frac{n\pi x}{\ell}$ and then integrate both sides from 0 to 2ℓ

$$I=\int_0^{2\ell} f(x)\cos\frac{n\pi x}{\ell}dx=\frac{a_o}{2}\int_0^{2\ell}\cos\frac{n\pi x}{\ell}dx+a_1\int_0^{2\ell}\cos\frac{\pi x}{\ell}\cos\frac{n\pi x}{\ell}dx+a_2\int_0^{2\ell}\cos\frac{2\pi x}{\ell}\cos\frac{n\pi x}{\ell}dx$$
$$+\ldots\ldots+a_n\int_0^{2\ell}\cos\frac{n\pi x}{\ell}\cos\frac{n\pi x}{\ell}dx$$
$$+b_1\int_0^{2\ell}\sin\frac{\pi x}{\ell}\cos\frac{n\pi x}{\ell}dx+b_2\int_0^{2\ell}\sin\frac{2\pi x}{\ell}\cos\frac{n\pi x}{\ell}dx$$
$$+\ldots\ldots+b_n\int_0^{2\ell}\sin\frac{n\pi x}{\ell}\cos\frac{n\pi x}{\ell}dx$$

$$\int_0^{2\ell} f(x)\cos\frac{n\pi x}{\ell}dx=a_n\ell$$

$$\therefore \boxed{a_n=\frac{1}{\ell}\int_0^{2\ell} f(x)\cos\frac{n\pi x}{\ell}dx=\frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx}$$

(2) $b_n=?$ to find b_n multiply eq(1) by $\sin\frac{n\pi x}{\ell}$ and then integrate both sides from 0 to 2ℓ

$$\therefore \boxed{b_n=\frac{1}{\ell}\int_0^{2\ell} f(x)\sin\frac{n\pi x}{\ell}dx=\frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx}$$

Example 19 : Find the Fourier series for the periodic function defined by :

(a) $f(x) = x \qquad 0 < x < 3$

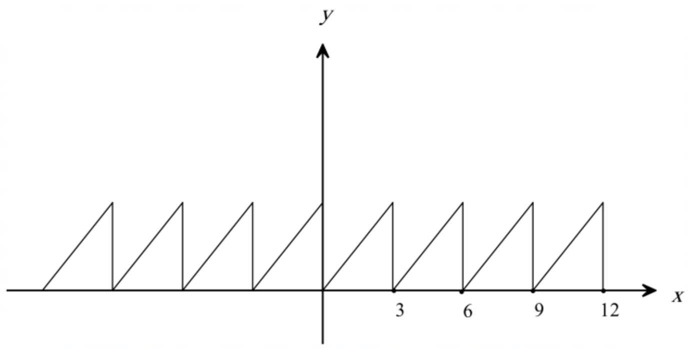
(b) $f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$

(c) $f(\theta) = \theta^2 \qquad 0 < \theta < 2\pi$

(d) $f(x) = \begin{cases} -x & -2 < x < 0 \\ x & 0 < x < 2 \end{cases}$

(e) $f(x) = x \qquad -2 < x < 2$

Solution: (a) $f(x) = x \qquad 0 < x < 3$



$$2\ell = 3 \qquad \Rightarrow \qquad \ell = \frac{3}{2}$$

$$f(x) = x$$

$$a_o = \frac{1}{\ell}\int_0^{2\ell} f(x)dx = \frac{1}{\frac{3}{2}}\int_0^3 x\,dx = \frac{2}{3}\left(\frac{x^2}{2}\Big|_0^3\right) = 3$$

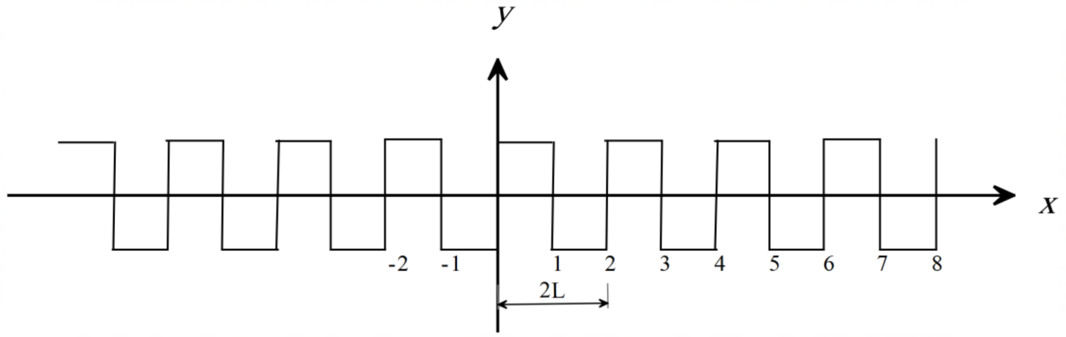
$$a_n = \frac{1}{\ell}\int_0^{2\ell} f(x)\cos\frac{n\pi x}{\ell}dx = \frac{2}{3}\int_0^3 x\cos\frac{n\pi x}{\ell}dx = 0$$

$$b_n = \frac{1}{\ell}\int_0^{2\ell} f(x)\sin\frac{n\pi x}{\ell}dx = \frac{2}{3}\int_0^3 x\sin\frac{n\pi x}{\ell}dx = -\frac{6}{n\pi}$$

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{\ell}$$

$$f(x) = \frac{3}{2} - \frac{6}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{\ell}$$

Solution: (b) $f(x)=\begin{cases}1 & 0<x<1\\ 0 & 1<x<2\end{cases}$



$$2\ell=2 \qquad \Rightarrow \qquad \ell=1$$

$$f(x)=1$$

$$a_o=\frac{1}{\ell}\int_0^{2\ell} f(x)dx=\frac{1}{1}\left(\int_0^1 1\,dx+\int_1^2 0dx\right)=\left(x\big|_0^1\right)=1$$

$$a_n=\frac{1}{\ell}\int_0^{2\ell} f(x)\cos\frac{n\pi x}{\ell}dx=\frac{1}{1}\left(\int_0^1 1\times\cos\frac{n\pi x}{1}dx+\int_1^2 0\times\cos\frac{n\pi x}{1}dx\right)$$

$$=\frac{1}{n\pi}\sin n\pi x\big|_0^1=\frac{1}{n\pi}[\sin n\pi-\sin 0]=\frac{1}{n\pi}[0-0]=0$$

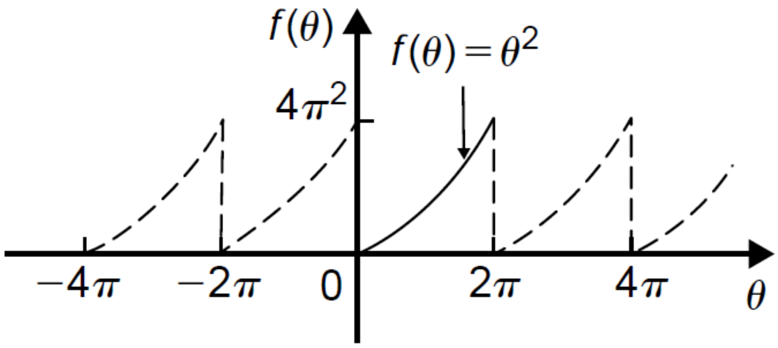
$$b_n=\frac{1}{\ell}\int_0^{2\ell} f(x)\sin\frac{n\pi x}{\ell}dx=\frac{1}{1}\left(\int_0^1 1\times\sin\frac{n\pi x}{1}dx+\int_1^2 0\times\sin\frac{n\pi x}{1}dx\right)$$

$$=-\frac{1}{n\pi}\cos n\pi x\big|_0^1=-\frac{1}{n\pi}[\cos n\pi-\cos 0]=\frac{1}{n\pi}[1-\cos n\pi]$$

$$f(x)=\frac{a_o}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{\ell}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{\ell}$$

$$f(x)=\frac{1}{2}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}[1-\cos n\pi]\sin n\pi x$$

Solution: (c) $f(\theta)=\theta^2 \qquad 0<\theta<2\pi$



$2\ell = 2\pi \qquad \Rightarrow \qquad \ell = \pi$

$f(x)=x$

$$a_o=\frac{1}{\ell}\int_0^{2\ell} f(\theta)d\theta=\frac{1}{\pi}\int_0^{2\pi}\theta^2 dx=\frac{1}{\pi}\left(\frac{\theta^3}{3}\Big|_0^{2\pi}\right)=\frac{8\pi^2}{3}$$

$$a_n=\frac{1}{\ell}\int_0^{2\ell} f(\theta)\cos\frac{n\pi\theta}{\ell}d\theta=\frac{1}{\pi}\int_0^{2\pi}\theta^2\cos\frac{n\pi\theta}{\pi}d\theta=\frac{1}{\pi}\int_0^{2\pi}\theta^2\cos n\theta d\theta$$

$$=\frac{1}{\pi n^3}\left[\theta^2 n^2\sin n\theta+2\theta n\cos n\theta-\sin n\theta\right]_0^{2\pi}$$

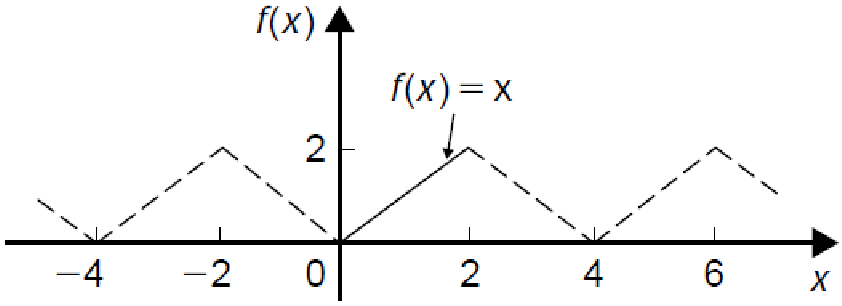
$$=\frac{1}{\pi n^3}\left[4(n\pi)^2\sin 2n\pi+4\pi n\cos 2n\pi-\sin 2n\pi\right]=\frac{4}{n^2}$$

$$b_n=\frac{1}{\ell}\int_0^{2\ell} f(\theta)\sin\frac{n\pi\theta}{\ell}d\theta=\frac{1}{\pi}\int_0^{2\pi}\theta^2\sin\frac{n\pi\theta}{\pi}d\theta=\frac{1}{\pi}\int_0^{2\pi}\theta^2\sin n\theta d\theta=-\frac{4\pi}{n}$$

$$f(x)=\frac{a_o}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{\ell}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{\ell}$$

$$f(x)=\frac{4\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx-4\pi\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

Solution: (d) $f(x)=\begin{cases}-x & -2<x<0\\ x & 0<x<2\end{cases}$



$2\ell=4 \qquad \Rightarrow \qquad \ell=2$

$f(x)=x$

$$a_o=\frac{1}{\ell}\int_0^{2\ell} f(x)dx=\frac{1}{2}\left(\int_0^2 x\,dx+\int_2^4 -x\,dx\right)=\frac{1}{2}\left(\frac{x^2}{2}\Big|_0^2-\frac{x^2}{2}\Big|_2^4\right)=-2$$

$$a_n=\frac{1}{\ell}\int_0^{2\ell} f(x)\cos\frac{n\pi x}{\ell}dx=\frac{1}{2}\left(\int_0^2 x\cos\frac{n\pi x}{2}dx-\int_2^4 x\cos\frac{n\pi x}{2}dx\right)$$

$$=\frac{1}{2}\left(\frac{2}{n\pi}x\sin\frac{n\pi x}{2}-\left(\frac{2}{n\pi}\right)^2\cos\frac{n\pi x}{2}\right)\Bigg|_0^2-\frac{1}{2}\left(\frac{2}{n\pi}x\sin\frac{n\pi x}{2}-\left(\frac{2}{n\pi}\right)^2\cos\frac{n\pi x}{2}\right)\Bigg|_2^4=\left(\frac{2}{n\pi}\right)^2$$

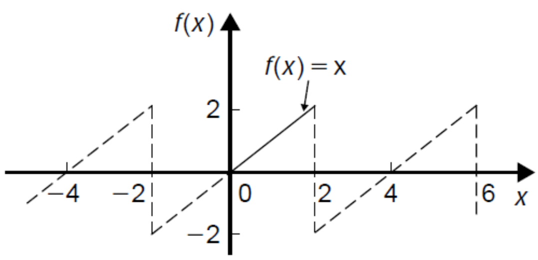
$$b_n==\frac{1}{\ell}\int_0^{2\ell} f(x)\sin\frac{n\pi x}{\ell}dx=\frac{1}{2}\left(\int_0^2 x\sin\frac{n\pi x}{2}dx-\int_2^4 x\sin\frac{n\pi x}{2}dx\right)$$

$$=\frac{1}{2}\left(-\frac{2}{n\pi}x\cos\frac{n\pi x}{2}+\left(\frac{2}{n\pi}\right)^2\sin\frac{n\pi x}{2}\right)\Bigg|_0^2-\frac{1}{2}\left(-\frac{2}{n\pi}x\cos\frac{n\pi x}{2}+\left(\frac{2}{n\pi}\right)^2\sin\frac{n\pi x}{2}\right)\Bigg|_2^4=\frac{8}{n\pi}$$

$$f(x)=\frac{a_o}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{\ell}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{\ell}$$

$$f(x)=-2+\left(\frac{2}{\pi}\right)^2\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{2}+\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2}$$

Solution: (e) $f(x)=x \qquad -2<x<2$ **H.W**



H.W 14 : Find the Fourier series for the periodic function defined by :

(a) $f(x)=\begin{cases} 0 & -2<x<0 \\ 1 & 0<x<1 \\ 2 & 1<x<2 \end{cases}$

(b) $f(x)=\begin{cases} 0 & -\pi<x<0 \\ x^2 & 0<x<\pi \end{cases}$

(c) $f(x)=\begin{cases} x^2 & 0<x<\pi \\ \pi & \pi<x<2\pi \end{cases}$

(d) $f(x)=\begin{cases} -x & -5<x<0 \\ 1+x^2 & 0<x<5 \end{cases}$

(e) $f(x)=\begin{cases} 2x & -3<x<-2 \\ 0 & -2<x<1 \\ x^2 & 1<x<3 \end{cases}$

(f) $f(x)=\begin{cases} \cos x & -2<x<0 \\ \sin x & 0<x<2 \end{cases}$

(g) $f(x)=\begin{cases} 1 & 0<x<1 \\ 0 & 1<x<3 \\ -1 & 3<x<5 \end{cases}$

(h) $f(x)=\begin{cases} -2 & -4<x<-2 \\ 1+x^2 & -2<x<2 \\ 0 & 2<x<4 \end{cases}$

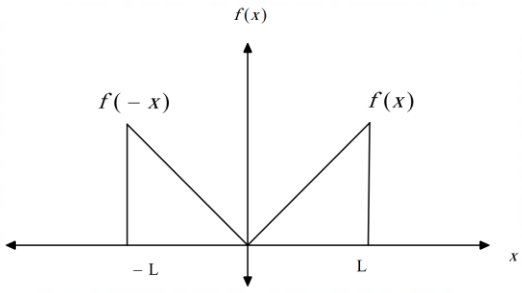
(i) $f(x)=\begin{cases} e^x-x & -\pi<x<0 \\ e^x+x & 0<x<\pi \end{cases}$

(j) $f(x)=\begin{cases} 0 & -\pi<x<-\frac{\pi}{2} \\ \pi+2x & -\frac{\pi}{2}<x<0 \\ \pi+2x & 0<x<\frac{\pi}{2} \\ 0 & \frac{\pi}{2}<x<\pi \end{cases}$

9.1- Even and Odd function

1- Even function f(x) = f(-x)

Ex : x^{2n} , cos(nx)



$$a_o = \frac{1}{\ell}\left[\int_{-\ell}^{0} f(-x)dx + \int_{0}^{\ell} f(x)dx\right]$$

$$= \frac{1}{\ell}\left[-\int_{0}^{-\ell} f(-x)dx + \int_{0}^{\ell} f(x)dx\right] = \frac{1}{\ell}\left[\int_{0}^{\ell} f(x)dx + \int_{0}^{\ell} f(x)dx\right] = \boxed{\frac{2}{\ell}\int_{0}^{\ell} f(x)dx}$$

$$a_n = \frac{1}{\ell}\left[\int_{-\ell}^{0} f(-x)\cos\frac{-n\pi x}{\ell}dx + \int_{0}^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx\right]$$

$$= \frac{1}{\ell}\left[\int_{0}^{-\ell} -f(x)\cos\frac{n\pi x}{\ell}dx + \int_{0}^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx\right]$$

$$= \frac{1}{\ell}\left[\int_{0}^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx + \int_{0}^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx\right] = \boxed{\frac{2}{\ell}\int_{0}^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx}$$

$$b_n = \frac{1}{\ell}\left[\int_{-\ell}^{0} f(-x)\sin\frac{-n\pi x}{\ell}dx + \int_{0}^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx\right]$$

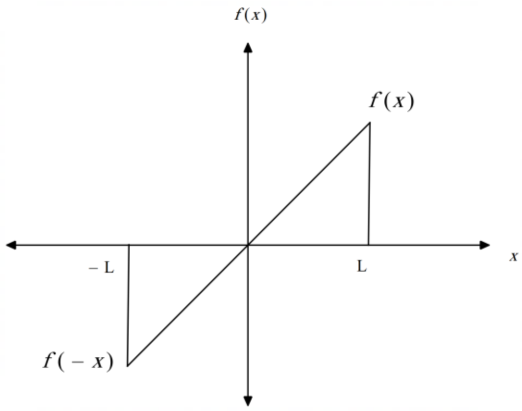
$$= \frac{1}{\ell}\left[\int_{-\ell}^{0} -f(x)\sin\frac{n\pi x}{\ell}dx + \int_{0}^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx\right] = \frac{1}{\ell}\left[\int_{0}^{-\ell} f(x)\sin\frac{n\pi x}{\ell}dx + \int_{0}^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx\right]$$

$$= \frac{1}{\ell}\left[-\int_{0}^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx + \int_{0}^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx\right] = \boxed{0}$$

$$\boxed{\therefore\ f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell}}$$

2- Odd function f(-x) = -f(x)

Ex : x^{2n+1} , sin(nx)



$$a_o = \frac{1}{\ell}\left[\int_{-\ell}^{0} f(-x)dx + \int_{0}^{\ell} f(x)dx\right]$$

$$= \frac{1}{\ell}\left[-\int_{-\ell}^{0} f(x)dx + \int_{0}^{\ell} f(x)dx\right] = \frac{1}{\ell}\left[\int_{0}^{-\ell} f(x)dx + \int_{0}^{\ell} f(x)dx\right]$$

$$= \frac{1}{\ell}\left[-\int_{0}^{\ell} f(x)dx + \int_{0}^{\ell} f(x)dx\right] = \boxed{0}$$

$$a_n = \frac{1}{\ell}\left[\int_{-\ell}^{0} f(-x)\cos\frac{-n\pi x}{\ell}dx + \int_{0}^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx\right] = \frac{1}{\ell}\left[\int_{-\ell}^{0} -f(x)\cos\frac{n\pi x}{\ell}dx + \int_{0}^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx\right]$$

$$= \frac{1}{\ell}\left[\int_{0}^{-\ell} f(x)\cos\frac{n\pi x}{\ell}dx + \int_{0}^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx\right]$$

$$= \frac{1}{\ell}\left[-\int_{0}^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx + \int_{0}^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx\right] = \boxed{0}$$

$$b_n = \frac{1}{\ell}\left[\int_{-\ell}^{0} f(-x)\sin\frac{-n\pi x}{\ell}dx + \int_{0}^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx\right]$$

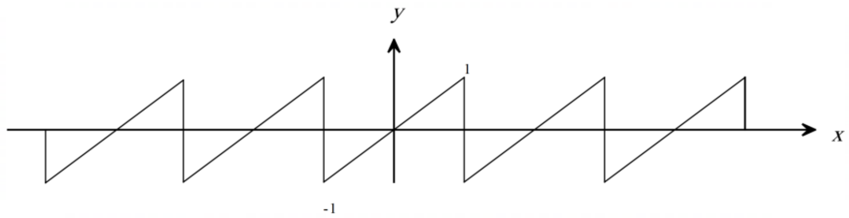
$$= \frac{1}{\ell}\left[\int_{-\ell}^{0} f(x)\sin\frac{n\pi x}{\ell}dx + \int_{0}^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx\right] = \frac{1}{\ell}\left[-\int_{0}^{-\ell} f(x)\sin\frac{n\pi x}{\ell}dx + \int_{0}^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx\right]$$

$$= \frac{1}{\ell}\left[\int_{0}^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx + \int_{0}^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx\right] = \boxed{\frac{2}{\ell}\int_{0}^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx}$$

$$\boxed{\therefore\ f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}}$$

Example 20 : Expand in Fourier series for the whose function in one period is f(x)=x -1<x<1

Solution: (a) $f(x) = x \qquad -1 < x < 1$



The function is odd $\quad a_o = 0 \quad a_n = 0$

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}$$

$$\ell = 1$$

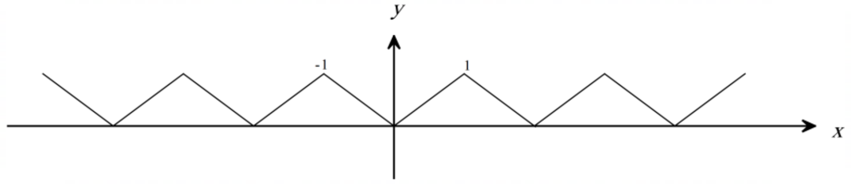
$$b_n = \frac{2}{1}\int_0^{1} x\sin\frac{n\pi x}{1}dx = 2\int_0^{1} x\sin(n\pi x)dx$$

$$= 2\left[\frac{-x}{n\pi}\cos n\pi x + \frac{1}{(n\pi)^2}\sin n\pi x\right]_0^1 = \frac{-2}{n\pi}\cos n\pi$$

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{-2}{n\pi}\cos n\pi \times \sin\frac{n\pi x}{1}\right) = \frac{-2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n}\cos n\pi \times \sin n\pi x\right)$$

Example 21 : Expand in Fourier series for the whose function in one period is

Solution: (a) $f(x) = |x| \qquad -1 < x < 1$



The function is even $\qquad b_n = 0$

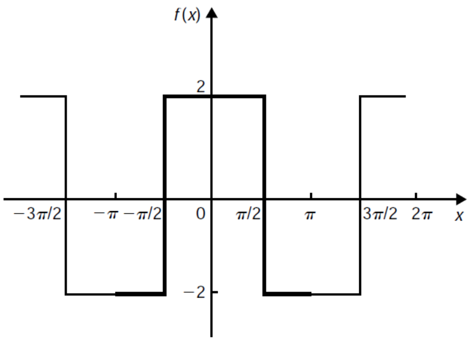
$$a_o = \frac{2}{\ell}\int_0^{\ell} f(x)dx = \frac{2}{1}\int_0^1 x\,dx = 2\left(\frac{x^2}{2}\right)_0^1 = 1$$

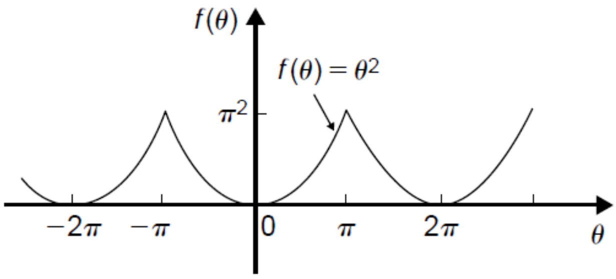
$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx = \frac{2}{1}\int_0^1 x\cos\frac{n\pi x}{1}dx = 2\int_0^1 x\cos n\pi x\,dx$$

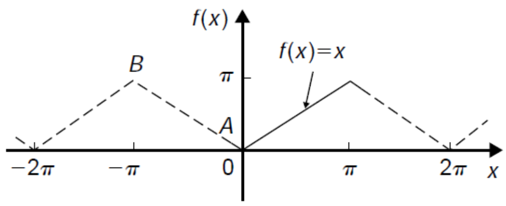
$$= 2\left(\frac{x}{n\pi}\sin n\pi x + \frac{1}{(n\pi)^2}\cos n\pi x\right)_0^1 = \frac{2}{(n\pi)^2}[\cos n\pi - 1]$$

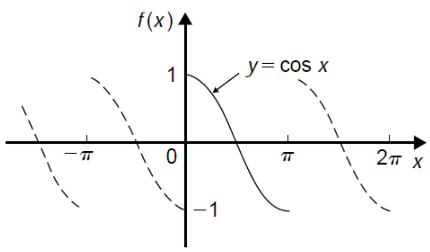
$$\therefore\ f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{\ell} = \frac{1}{2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left([\cos n\pi - 1]\times\cos\frac{n\pi x}{\ell}\right)$$

H.W 15 : Find the Fourier series for the periodic function defined by :



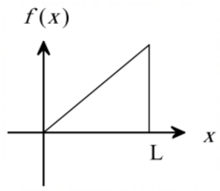






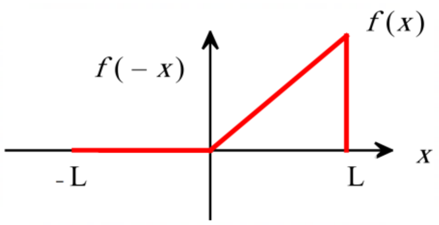
9.2- Half – Range Fourier Series

When f(x) is define for $0 \le x \le \ell$. Required to define the function f(x) for $-\ell \le x \le 0$



9.2.1 Case 1

$$f(-x) = 0 \qquad -\ell \le x \le 0$$
$$f(x) = f(1) \qquad 0 \le x \le \ell$$



$$a_o = \frac{1}{\ell}\left[\int_{-\ell}^{0}(0)dx + \int_{0}^{\ell} f(x)dx\right] = \frac{1}{\ell}\int_{0}^{\ell} f(x)dx$$

$$a_n = \frac{1}{\ell}\left[\int_{-\ell}^{0}(0)\cos\frac{-n\pi x}{\ell}dx + \int_{0}^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx\right] = \frac{1}{\ell}\int_{0}^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx$$

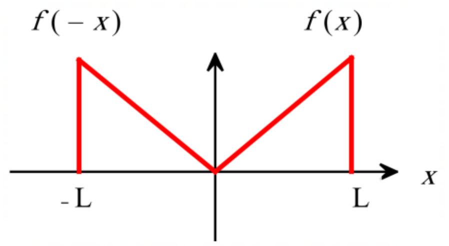
$$b_n = \frac{1}{\ell}\left[\int_{-\ell}^{0}(0)\sin\frac{-n\pi x}{\ell}dx + \int_{0}^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx\right] = \frac{1}{\ell}\int_{0}^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx$$

$$\therefore\ f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}$$

9.2.2 Case 2

$f(-x)=f(x) \qquad -\ell \le x \le 0$

$f(x)=f(-x) \qquad 0 \le x \le \ell$



$$a_o = \frac{2}{\ell}\int_0^{\ell} f(x)dx$$

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx$$

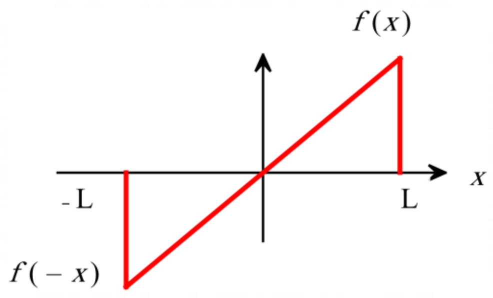
$$b_n = 0$$

$$\therefore \; f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell}$$

9.2.3 Case 3

$f(-x)=-f(x) \qquad -\ell \le x \le 0$

$f(x)=f(-x) \qquad 0 \le x \le \ell$



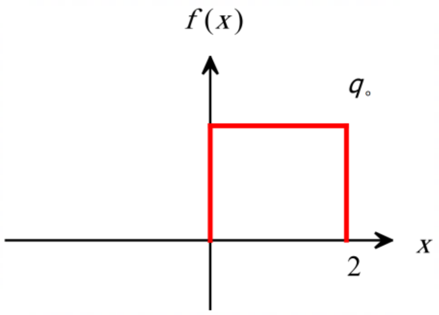
The function is odd $\qquad a_o=0 \qquad a_n=0$

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}$$

Example 22 : Calculate the half-range Fourier series for $q_o \quad 0 \le x \le 2$ where is constant q_o

Solution:



$$a_o = \frac{1}{\ell}\int_0^{\ell} f(x)dx = \frac{1}{2}\int_0^{2}(q_o)dx = \frac{1}{2}(q_o x)\Big|_0^2 = q_o$$

$$a_n = \frac{1}{2}\int_0^{2} q_o \cos\frac{n\pi x}{2}dx = \frac{q_o}{n\pi}\left(\sin\frac{n\pi x}{2}\right)\Bigg|_0^2 = \frac{q_o}{n\pi}(\sin n\pi - \sin 0) = 0$$

$$b_n = \frac{1}{2}\int_0^{2} q_o \sin\frac{n\pi x}{2}dx = -\frac{q_o}{n\pi}\left(\cos\frac{n\pi x}{2}\right)\Bigg|_0^2 = -\frac{q_o}{n\pi}(\cos n\pi - \cos 0) = \frac{2q_o}{n\pi}$$

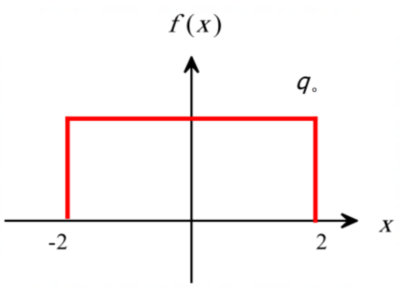
$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}$$

$$f(x) = \frac{q_o}{2} + \frac{2q_o}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2}$$

$$f(x) = \frac{q_o}{2} + \frac{2q_o}{\pi}\left[\sin\frac{\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \frac{1}{5}\sin\frac{5\pi x}{2} + \frac{1}{7}\sin\frac{7\pi x}{2} + \ldots\right]$$

Example 23 : Calculate the half-range Fourier series for $q_o \quad -2 \le x \le 2$ where is constant q_o

Solution:



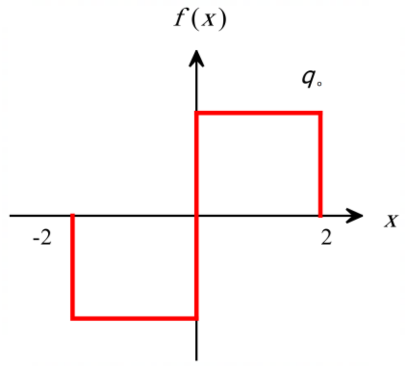
$$a_o = \frac{2}{\ell}\int_0^{\ell} f(x)dx = \frac{2}{2}\int_0^{2}(q_o)dx = 2q_o$$

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx = \frac{2}{2}\int_0^{2}(q_o)\cos\frac{n\pi x}{2}dx = 0$$

$$f(x) = \frac{a_o}{2} = \frac{2q_o}{2} = q_o$$

Example 24 : Calculate the half-range Fourier series for $q_o \quad -2 \le x \le 2$ where is constant q_o

Solution:



The function is odd $\quad a_o=0 \quad a_n=0$

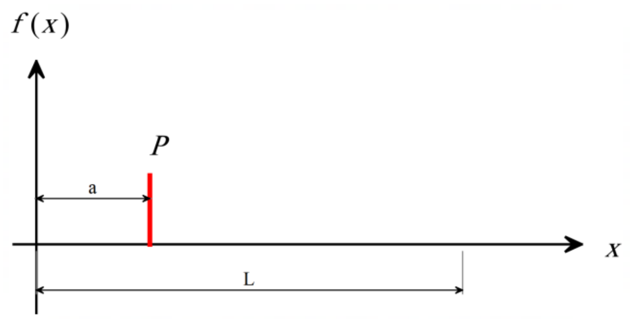
$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx == \frac{2}{2}\int_0^{2}(q_o)\sin\frac{n\pi x}{2}dx$$

$$= -\frac{2q_o}{n\pi}\left(\cos\frac{n\pi x}{2}\right)\Bigg|_0^2 = -\frac{2q_o}{n\pi}(\cos n\pi - \cos 0) = \frac{4q_o}{n\pi}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell} = \frac{4q_o}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2}$$

$$= \frac{4q_o}{\pi}\left[\sin\frac{\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \frac{1}{5}\sin\frac{5\pi x}{2} + \frac{1}{7}\sin\frac{7\pi x}{2} + \dots\right]$$

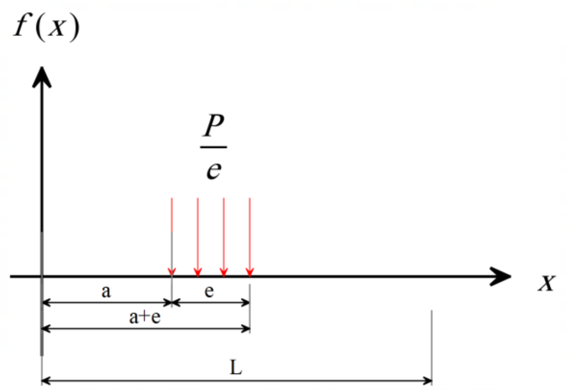
9.1- Impulsive function



$$f(x)=\begin{cases} P & x=a \\ 0 & elsewhere \end{cases}$$

$$or \qquad f(x)=P\ \delta(x-a)$$

Expand in half-range sine series



The function is odd $\quad a_o=0 \quad a_n=0$

$$b_n=\frac{2}{\ell}\int_a^{a+e}\frac{P}{e}\sin\frac{n\pi x}{\ell}dx=\frac{2P}{\ell}\left(-\frac{\ell}{n\pi e}\cos\frac{n\pi x}{\ell}\right)\Bigg|_a^{a+e}=-\frac{2P}{n\pi e}\left(\cos\frac{n\pi(a+e)}{\ell}-\cos\frac{n\pi a}{\ell}\right)$$

$$=-\frac{2P}{n\pi e}\left(\cos\frac{n\pi a}{\ell}\cos\frac{n\pi e}{\ell}-\sin\frac{n\pi a}{\ell}\sin\frac{n\pi e}{\ell}-\cos\frac{n\pi a}{\ell}\right)$$

$$When \qquad e\to 0 \qquad \cos\frac{n\pi e}{\ell}=1 \qquad \sin\frac{n\pi e}{\ell}=\frac{n\pi e}{\ell}$$

$$=-\frac{2P}{n\pi e}\left(\cancel{\cos\frac{n\pi a}{\ell}}-\frac{n\pi e}{\ell}\sin\frac{n\pi a}{\ell}-\cancel{\cos\frac{n\pi a}{\ell}}\right)$$

$$=-\frac{2P}{\cancel{n\pi e}}\left(-\frac{\cancel{n\pi e}}{\ell}\sin\frac{n\pi a}{\ell}\right)=\frac{2P}{\ell}\sin\frac{n\pi a}{\ell}$$

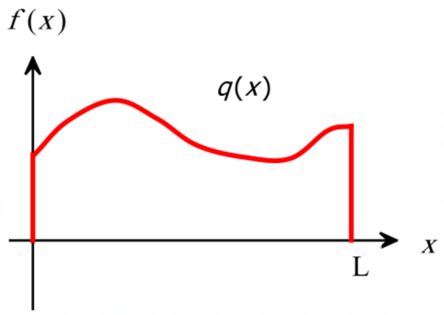
$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{\ell}$$

$$f(x)=\frac{2P}{\ell}\sum_{n=1}^{\infty}\sin\frac{n\pi a}{\ell}\sin\frac{n\pi x}{\ell}$$

10- Applications of Fourier Series

10.1- Deflection equation of simply supported beam

$$\frac{d^4y}{dx^4}=\frac{q(x)}{EI}$$



Note

$$\frac{d^2y}{dx^2}=\frac{M}{EI} \quad Moment$$

$$\frac{d^3y}{dx^3}=\frac{V}{EI} \quad Shear$$

$$\frac{d^4y}{dx^4}=\frac{W}{EI} \quad Load$$

Expand in half-range sine series

$$q(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell} \qquad b_n=\frac{2}{\ell}\int_0^{\ell} q(x)\sin\frac{n\pi x}{\ell}dx$$

$$Let \quad y(x)=\sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{\ell}$$

to satisfy the boundary conditions $y=0 \quad at \quad x=0\ \&\ \ell$

$$y(0)=\sum_{n=1}^{\infty} C_n \sin\frac{n\pi(0)}{\ell}=0 \quad and \quad y(\ell)=\sum_{n=1}^{\infty} C_n \sin\frac{n\pi(\ell)}{\ell}=0 \quad O.K.$$

$$\because \quad \frac{d^4y}{dx^4}=\frac{q(x)}{EI}$$

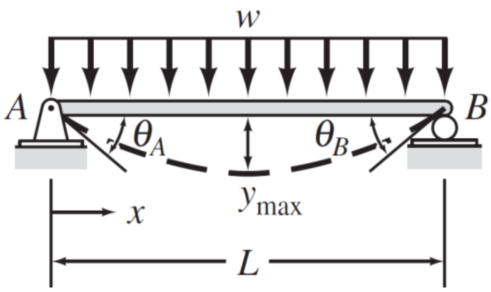
$$\because \quad \frac{d^4y}{dx^4}=\sum_{n=1}^{\infty} C_n\left(\frac{n\pi}{\ell}\right)^4 \sin\frac{n\pi x}{\ell}$$

$$\therefore \quad \sum_{n=1}^{\infty} C_n\left(\frac{n\pi}{\ell}\right)^4 \sin\frac{n\pi x}{\ell}=\frac{\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}}{EI}$$

$$\sum_{n=1}^{\infty} C_n\left(\frac{n\pi}{\ell}\right)^4 \sin\frac{n\pi x}{\ell}=\sum_{n=1}^{\infty}\frac{b_n}{EI}\sin\frac{n\pi x}{\ell}$$

$$C_n\left(\frac{n\pi}{\ell}\right)^4=\frac{b_n}{EI} \qquad \therefore \boxed{C_n=\frac{b_n}{EI}\left(\frac{\ell}{n\pi}\right)^4}$$

Example 25 : Find the deflection equation for simply supported beam with uniform distribution load using Fourier series method.



Solution :

$$y(x)=\sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{\ell} \qquad C_n=\frac{b_n}{EI}\left(\frac{\ell}{n\pi}\right)^4$$

$$b_n=\frac{2}{\ell}\int_0^{\ell} q(x)\sin\frac{n\pi x}{\ell}dx \qquad q(x)=q$$

$$b_n=\frac{2}{\ell}\int_0^{\ell} q\sin\frac{n\pi x}{\ell}dx=\frac{2q}{\ell}\int_0^{\ell}\sin\frac{n\pi x}{\ell}dx=\frac{2q}{\ell}\left(-\frac{\ell}{n\pi}\cos\frac{n\pi x}{\ell}\right)\Bigg|_0^{\ell}$$

$$=-\frac{2q}{n\pi}\left(\cos\frac{n\pi\ell}{\ell}-\cos\frac{n\pi\times 0}{\ell}\right)=-\frac{2q}{n\pi}(\cos n\pi-\cos 0)=\frac{2q}{n\pi}(1-\cos n\pi)$$

$$=\frac{2q}{n\pi}(1-(-1))=\frac{4q}{n\pi} \qquad n=1,3,5,7,9,\ldots\ldots\ldots\infty \qquad Odd-Function$$

$$\therefore \qquad C_n=\frac{b_n}{EI}\left(\frac{\ell}{n\pi}\right)^4=C_n=\frac{4q}{EI}\frac{\ell^4}{(n\pi)^5}$$

$$\therefore \qquad y(x)=\sum_{n=1}^{\infty} C_n\sin\frac{n\pi x}{\ell}=\boxed{\frac{4q\ell^4}{\pi^5 EI}\sum_{n=1}^{\infty}\frac{1}{n^5}\sin\frac{n\pi x}{\ell}}$$

at Mid-Span $\qquad x=\frac{\ell}{2} \qquad y_{exact}=\frac{5q\ell^4}{384EI}=0.013020833333\frac{q\ell^4}{EI}$

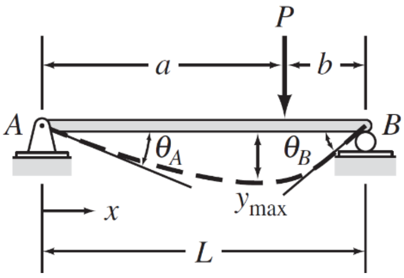
$$y\left(x=\frac{\ell}{2}\right)=\frac{4q\ell^4}{\pi^5 EI}\sum_{n=1}^{\infty}\frac{1}{n^5}\sin\frac{n\pi(\frac{\ell}{2})}{\ell}=\frac{4q\ell^4}{\pi^5 EI}\sum_{n=1}^{\infty}\frac{1}{n^5}\sin\frac{n\pi}{2}$$

$n=1 \qquad y=0.01307105457\frac{q\ell^4}{EI} \qquad Error=0.3857\%$

$n=3 \qquad y=0.01301726422\frac{q\ell^4}{EI} \qquad Error=0.0274\%$

$n=21 \qquad y=0.01302083456\frac{q\ell^4}{EI} \qquad Error\simeq 0.0\%$

Example 26 : Find the deflection equation for simply supported beam with concentrated load using Fourier series method.



Solution :

$$q(x) = P \quad at\ x = a \qquad b_n = \frac{2}{\ell}\int_0^{\ell} P\sin\frac{n\pi x}{\ell}dx = \frac{2P}{\ell}\int_a^a \sin\frac{n\pi x}{\ell}dx = 0$$

Distribute P over small distance e

$$b_n = \frac{2}{\ell}\int_a^{a+e}\frac{P}{e}\sin\frac{n\pi x}{\ell}dx = \frac{2P}{\ell}\left(-\frac{\ell}{n\pi e}\cos\frac{n\pi x}{\ell}\right)\Bigg|_a^{a+e} = -\frac{2P}{n\pi e}\left(\cos\frac{n\pi(a+e)}{\ell} - \cos\frac{n\pi a}{\ell}\right)$$

$$= -\frac{2P}{n\pi e}\left(\cos\frac{n\pi a}{\ell}\cos\frac{n\pi e}{\ell} - \sin\frac{n\pi a}{\ell}\sin\frac{n\pi e}{\ell} - \cos\frac{n\pi a}{\ell}\right)$$

$$When \quad e \to 0 \qquad \cos\frac{n\pi e}{\ell} = 1 \qquad \sin\frac{n\pi e}{\ell} = \frac{n\pi e}{\ell}$$

$$= -\frac{2P}{n\pi e}\left(\cancel{\cos\frac{n\pi a}{\ell}} - \frac{n\pi e}{\ell}\sin\frac{n\pi a}{\ell} - \cancel{\cos\frac{n\pi a}{\ell}}\right) = -\frac{2P}{\cancel{n\pi e}}\left(-\frac{\cancel{n\pi e}}{\ell}\sin\frac{n\pi a}{\ell}\right) = \frac{2P}{\ell}\sin\frac{n\pi a}{\ell}$$

$$\therefore \quad C_n = \frac{b_n}{EI}\left(\frac{\ell}{n\pi}\right)^4 = C_n = \frac{2P}{EI}\frac{\ell^3}{(n\pi)^4}\sin\frac{n\pi a}{\ell}$$

$$\therefore \quad y(x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{\ell} = \boxed{\frac{2P\ell^3}{\pi^4 EI}\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi a}{\ell}\sin\frac{n\pi x}{\ell}}$$

$$if \quad a = \frac{\ell}{2} \quad \text{at Mid-Span} \quad x = \frac{\ell}{2} \qquad y_{exact} = \frac{P\ell^3}{48EI} = 0.020833333\frac{P\ell^3}{EI}$$

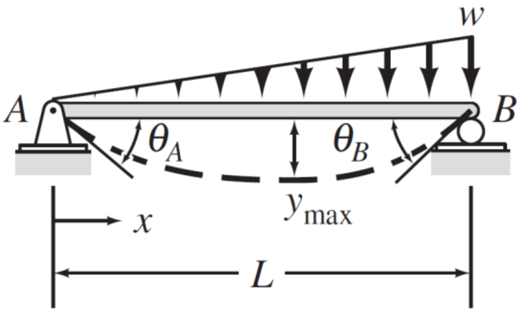
$$y\left(x = \frac{\ell}{2}\right) = \frac{2P\ell^3}{\pi^4 EI}\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi\frac{\ell}{2}}{\ell}\sin\frac{n\pi\frac{\ell}{2}}{\ell} = \frac{2P\ell^3}{\pi^4 EI}\sum_{n=1}^{\infty}\frac{1}{n^4}\sin^2\frac{n\pi}{2}$$

$$n = 1 \qquad y = 0.02053196451\frac{P\ell^3}{EI} \qquad Error = 1.4466\%$$

$$n = 3 \qquad y = 0.02078544555\frac{P\ell^3}{EI} \qquad Error = 0.2299\%$$

$$n = 21 \qquad y = 0.02083301328\frac{P\ell^3}{EI} \qquad Error \simeq 0.0\%$$

Example 27 : Find the deflection equation for simply supported beam with linear distribution load using Fourier series method.



Solution :

$$y(x)=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi x}{\ell} \qquad C_n=\frac{b_n}{EI}\left(\frac{\ell}{n\pi}\right)^4$$

$$b_n=\frac{2}{\ell}\int_0^{\ell}q(x)\sin\frac{n\pi x}{\ell}dx \qquad q(x)=q\frac{x}{\ell}$$

$$b_n=\frac{2}{\ell}\int_0^{\ell}q\frac{x}{\ell}\sin\frac{n\pi x}{\ell}dx=\frac{2q}{\ell^2}\int_0^{\ell}x\sin\frac{n\pi x}{\ell}dx=\frac{2q}{\ell^2}\left(\left(\frac{\ell}{n\pi}\right)^2\sin\frac{n\pi x}{\ell}-\frac{\ell}{n\pi}x\cos\frac{n\pi x}{\ell}\right)\Bigg|_0^{\ell}$$

$$=2q\left(-n\pi\cos n\pi\right)=\frac{2q}{n\pi}(-1)^{n+1} \qquad n=1,2,3,4,5,..........\infty$$

$$\therefore \qquad C_n=\frac{b_n}{EI}\left(\frac{\ell}{n\pi}\right)^4=C_n=\frac{2q\ell^4}{\pi^5EI}\frac{(-1)^{n+1}}{n^5}$$

$$\therefore \qquad y(x)=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi x}{\ell}=\boxed{\frac{2q\ell^4}{\pi^5EI}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^5}\sin\frac{n\pi x}{\ell}}$$

at Mid-Span $\qquad x=\frac{\ell}{2} \qquad y_{exact}=\frac{5q\ell^4}{768EI}=0.006510416\frac{q\ell^4}{EI}$

$$y\left(x=\frac{\ell}{2}\right)=\frac{2q\ell^4}{\pi^5EI}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^5}\sin\frac{n\pi(\frac{\ell}{2})}{\ell}=\frac{2q\ell^4}{\pi^5EI}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^5}\sin\frac{n\pi}{2}$$

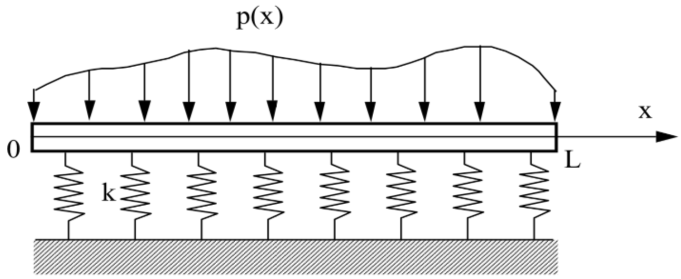
$n=1 \qquad y=0.00653552729\frac{q\ell^4}{EI} \qquad Error=0.3857\%$

$n=5 \qquad y=0.00651072348\frac{q\ell^4}{EI} \qquad Error=0.005\%$

10.2- Beam on Elastic Foundation

Simply supported beams on elastic foundation, ***k*** = subgrade soil reaction (spring constant)

Winkler Concept $k_{beam} = \dfrac{Load/Area}{Settlement} \times Width_{beam}$



$$\frac{d^4y}{dx^4} = \frac{q - ky}{EI}$$

$$\frac{d^4y}{dx^4} + \frac{k}{EI}y = \frac{q}{EI} \quad ------(1)$$

$$b_n = \frac{2}{\ell}\int_0^{\ell} q(x)\sin\frac{n\pi x}{\ell}dx$$

$$q(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell} \quad ------(2)$$

$$y(x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{\ell} \quad ------(3)$$

$$\frac{d^4y}{dx^4} = \sum_{n=1}^{\infty} C_n \left(\frac{n\pi}{\ell}\right)^4 \sin\frac{n\pi x}{\ell} \quad ------(4)$$

Sub 2,3 & 4 into 1

$$\sum_{n=1}^{\infty} C_n \left(\frac{n\pi}{\ell}\right)^4 \sin\frac{n\pi x}{\ell} + \frac{k}{EI}\sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{\ell} = \frac{1}{EI}\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}$$

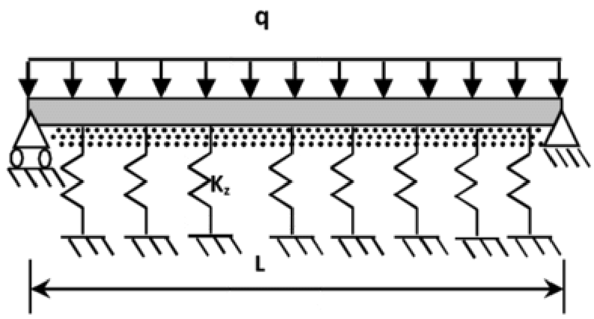
$$C_n \left(\frac{n\pi}{\ell}\right)^4 \cancel{\sum_{n=1}^{\infty}\sin\frac{n\pi x}{\ell}} + \frac{k}{EI}C_n \cancel{\sum_{n=1}^{\infty}\sin\frac{n\pi x}{\ell}} = \frac{1}{EI}b_n \cancel{\sum_{n=1}^{\infty}\sin\frac{n\pi x}{\ell}}$$

$$C_n \left[\left(\frac{n\pi}{\ell}\right)^4 + \frac{k}{EI}\right] = \frac{1}{EI}b_n \quad \Rightarrow$$

$$\boxed{C_n = \frac{b_n \ell^4}{(n\pi)^4 EI + \ell^4 k}}$$

Example 28 : Find the deflection equation (elastic curve) for simply supported beam on elastic foundation with uniform distribution load using Fourier series method.

Solution :



$$y(x)=\sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{\ell} \qquad C_n=\frac{b_n \ell^4}{(n\pi)^4 EI+\ell^4 k}$$

$$b_n=\frac{2}{\ell}\int_0^{\ell} q(x)\sin\frac{n\pi x}{\ell}dx \qquad q(x)=q$$

$$b_n=\frac{2}{\ell}\int_0^{\ell} q\sin\frac{n\pi x}{\ell}dx=\frac{2q}{\ell}\int_0^{\ell}\sin\frac{n\pi x}{\ell}dx=\frac{2q}{\ell}\left(-\frac{\ell}{n\pi}\cos\frac{n\pi x}{\ell}\right)\Bigg|_0^{\ell}$$

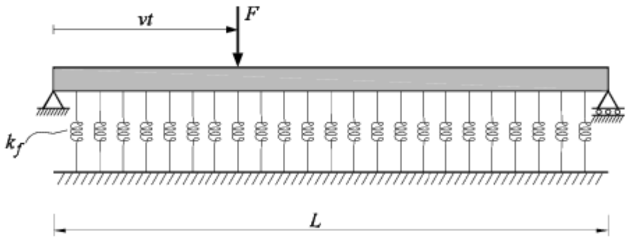
$$=-\frac{2q}{n\pi}\left(\cos\frac{n\pi\ell}{\ell}-\cos\frac{n\pi\times 0}{\ell}\right)=-\frac{2q}{n\pi}(\cos n\pi-\cos 0)=\frac{2q}{n\pi}(1-\cos n\pi)$$

$$=\frac{2q}{n\pi}(1-(-1))=\frac{4q}{n\pi} \qquad n=1,3,5,7,9,..........\infty \qquad Odd-Function$$

$$\therefore \quad C_n=\frac{b_n\ell^4}{(n\pi)^4 EI+\ell^4 k}=C_n=\frac{4q\ell^4}{(n\pi)^5 EI+n\pi\ell^4 k}$$

$$\therefore \quad y(x)=\sum_{n=1}^{\infty} C_n\sin\frac{n\pi x}{\ell}=\boxed{4q\ell^4\sum_{n=1}^{\infty}\frac{1}{(n\pi)^5 EI+n\pi\ell^4 k}\sin\frac{n\pi x}{\ell}}$$

Example 29 : Find the deflection equation (elastic curve) for simply supported beam on elastic foundation with concentrated load using Fourier series method.



Solution :

$$y(x)=\sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{\ell} \qquad C_n=\frac{b_n \ell^4}{(n\pi)^4 EI+\ell^4 k}$$

$$b_n=\frac{2}{\ell}\int_0^{\ell} q(x)\sin\frac{n\pi x}{\ell}dx \qquad q(x)=q$$

$$q(x)=P \quad at\ x=a \qquad b_n=\frac{2}{\ell}\int_0^{\ell} P\sin\frac{n\pi x}{\ell}dx=\frac{2P}{\ell}\int_a^{a}\sin\frac{n\pi x}{\ell}dx=0$$

Distribute P over small distance e

$$b_n=\frac{2}{\ell}\int_a^{a+e}\frac{P}{e}\sin\frac{n\pi x}{\ell}dx=\frac{2P}{\ell}\left(-\frac{\ell}{n\pi e}\cos\frac{n\pi x}{\ell}\right)\Bigg|_a^{a+e}=-\frac{2P}{n\pi e}\left(\cos\frac{n\pi(a+e)}{\ell}-\cos\frac{n\pi a}{\ell}\right)$$

$$=-\frac{2P}{n\pi e}\left(\cos\frac{n\pi a}{\ell}\cos\frac{n\pi e}{\ell}-\sin\frac{n\pi a}{\ell}\sin\frac{n\pi e}{\ell}-\cos\frac{n\pi a}{\ell}\right)$$

$$When \quad e\to 0 \qquad \cos\frac{n\pi e}{\ell}=1 \qquad \sin\frac{n\pi e}{\ell}=\frac{n\pi e}{\ell}$$

$$=-\frac{2P}{n\pi e}\left(\cancel{\cos\frac{n\pi a}{\ell}}-\frac{n\pi e}{\ell}\sin\frac{n\pi a}{\ell}-\cancel{\cos\frac{n\pi a}{\ell}}\right)=-\frac{2P}{\cancel{n\pi e}}\left(-\frac{\cancel{n\pi e}}{\ell}\sin\frac{n\pi a}{\ell}\right)=\frac{2P}{\ell}\sin\frac{n\pi a}{\ell}$$

$$\therefore \quad C_n=\frac{b_n\ell^4}{(n\pi)^4 EI+\ell^4 k}=C_n=\frac{2P\ell^3}{(n\pi)^5 EI+n\pi\ell^4 k}\sin\frac{n\pi a}{\ell}$$

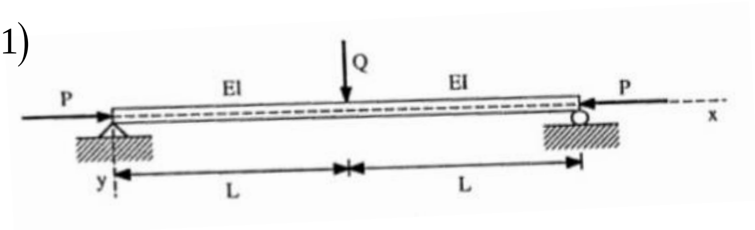
$$\therefore \quad y(x)=\sum_{n=1}^{\infty} C_n\sin\frac{n\pi x}{\ell}=\boxed{2P\ell^3\sum_{n=1}^{\infty}\frac{1}{(n\pi)^5 EI+n\pi\ell^4 k}\sin\frac{n\pi a}{\ell}\sin\frac{n\pi x}{\ell}}$$

$$if \quad a=\frac{\ell}{2}$$

$$\therefore \quad y(x)=2P\ell^3\sum_{n=1}^{\infty}\frac{1}{(n\pi)^5 EI+n\pi\ell^4 k}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{\ell}$$

10.3- Beam – Column Action

$$\frac{d^4y}{dx^4}+\frac{P}{EI}\frac{d^2y}{dx^2}=\frac{q}{EI} \qquad ------(1)$$



$$b_n=\frac{2}{\ell}\int_0^{\ell} q(x)\sin\frac{n\pi x}{\ell}dx$$

$$q(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell} \qquad ------(2)$$

$$y(x)=\sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{\ell}$$

$$\frac{d^2y}{dx^2}=-\sum_{n=1}^{\infty} C_n\left(\frac{n\pi}{\ell}\right)^2\sin\frac{n\pi x}{\ell} \qquad ------(3)$$

$$\frac{d^4y}{dx^4}=\sum_{n=1}^{\infty} C_n\left(\frac{n\pi}{\ell}\right)^4\sin\frac{n\pi x}{\ell} \qquad ------(4)$$

$Sub \quad 2,3\,\&\,4 \quad$ into 1

$$\sum_{n=1}^{\infty} C_n\left(\frac{n\pi}{\ell}\right)^4\sin\frac{n\pi x}{\ell}-\frac{P}{EI}\sum_{n=1}^{\infty} C_n\left(\frac{n\pi}{\ell}\right)^2\sin\frac{n\pi x}{\ell}=\frac{1}{EI}\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{\ell}$$

$$C_n\left(\frac{n\pi}{\ell}\right)^4\sum_{n=1}^{\infty}\sin\frac{n\pi x}{\ell}-\frac{P}{EI}C_n\left(\frac{n\pi}{\ell}\right)^2\sum_{n=1}^{\infty}\sin\frac{n\pi x}{\ell}=\frac{1}{EI}b_n\sum_{n=1}^{\infty}\sin\frac{n\pi x}{\ell}$$

$$C_n\left[\left(\frac{n\pi}{\ell}\right)^4-\frac{P}{EI}\left(\frac{n\pi}{\ell}\right)^2\right]=\frac{1}{EI}b_n$$

$$\boxed{C_n=\frac{b_n\ell^4}{(n\pi)^4EI-(n\pi\ell)^2P}}$$

Example 30 : Find the deflection equation (elastic curve) for simply supported beam with uniform distribution load and lateral force P action using Fourier series method.

Solution :

$$b_n=\frac{4q}{n\pi} \qquad C_n=\frac{b_n\ell^4}{(n\pi)^4EI-(n\pi\ell)^2P}=\frac{4q\ell^4}{(n\pi)^5EI-(n\pi)^3\ell^2P}$$

$$\therefore \quad y(x)=\sum_{n=1}^{\infty} C_n\sin\frac{n\pi x}{\ell}=\boxed{4q\ell^4\sum_{n=1}^{\infty}\frac{1}{(n\pi)^5EI-(n\pi)^3\ell^2P}\sin\frac{n\pi x}{\ell}}$$

10.4- Solution of differential equation

Example 31 : Use Half – Range sine series to solve the differential equation

$$\frac{d^2y}{dx^2}+y=10 \qquad y=0 \ at \qquad x=0,1$$

Solution :

$$\frac{d^2y}{dx^2}+y=10 \qquad -----(1)$$

$$y(x)=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi x}{\ell} \qquad -----(2)$$

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{\ell} \qquad -----(3)$$

$$\frac{d^2y}{dx^2}=-\sum_{n=1}^{\infty}C_n\left(\frac{n\pi}{\ell}\right)^2\sin\frac{n\pi x}{\ell} \qquad -----(4)$$

$$Sub \quad 2,3\,\&\,4 \quad \text{into } 1$$

$$-\sum_{n=1}^{\infty}C_n\left(\frac{n\pi}{\ell}\right)^2\sin\frac{n\pi x}{\ell}+\sum_{n=1}^{\infty}C_n\sin\frac{n\pi x}{\ell}=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{\ell}$$

$$-C_n\left(\frac{n\pi}{\ell}\right)^2\sum_{n=1}^{\infty}\sin\frac{n\pi x}{\ell}+C_n\sum_{n=1}^{\infty}\sin\frac{n\pi x}{\ell}=b_n\sum_{n=1}^{\infty}\sin\frac{n\pi x}{\ell}$$

$$C_n\left[1-\left(\frac{n\pi}{\ell}\right)^2\right]=b_n \qquad where \qquad \ell=1 \qquad \Rightarrow \qquad \boxed{C_n=\frac{b_n}{1-(n\pi)^2}}$$

$$b_n=\frac{2}{\ell}\int_0^{\ell}f(x)\sin\frac{n\pi x}{\ell}dx$$

$$b_n=\frac{2}{1}\int_0^{1}10\sin\frac{n\pi x}{1}dx=20\int_0^{1}\sin n\pi x\,dx=\frac{20}{\ell}\left(-\frac{1}{n\pi}\cos n\pi x\right)\Big|_0^1$$

$$=-\frac{20}{n\pi}(\cos n\pi-\cos 0)=\frac{20}{n\pi}(1-\cos n\pi)=\frac{20}{n\pi}(1-(-1))$$

$$=\frac{40}{n\pi} \quad n=1,3,5,7,9,.........\infty \qquad Odd-Function$$

$$y(x)=\sum_{n=1}^{\infty}\frac{b_n}{1-(n\pi)^2}\sin n\pi x=40\sum_{n=1}^{\infty}\frac{1}{n\pi-(n\pi)^3}\sin n\pi x$$

Example 31 : Use Half – Range Fourier series to solve the differential equation

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}=x \qquad y=0 \ at \qquad x=0,2$$

Solution :

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}=x \qquad -----(1)$$

$$y(x)=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi x}{\ell} \qquad -----(2) \qquad f(x)=\sum_{n=1}^{\infty}b_n\cos\frac{n\pi x}{\ell} \qquad -----(3)$$

$$\frac{dy}{dx}=-\sum_{n=1}^{\infty}C_n\left(\frac{n\pi}{\ell}\right)\cos\frac{n\pi x}{\ell} \qquad -----(4)$$

$$\frac{d^2y}{dx^2}=-\sum_{n=1}^{\infty}C_n\left(\frac{n\pi}{\ell}\right)^2\sin\frac{n\pi x}{\ell} \qquad -----(5)$$

$Sub \quad 3,4\,\&\,5 \quad \text{into } 1$

$$-\sum_{n=1}^{\infty}C_n\left(\frac{n\pi}{\ell}\right)^2\sin\frac{n\pi x}{\ell}-\sum_{n=1}^{\infty}C_n\left(\frac{n\pi}{\ell}\right)\cos\frac{n\pi x}{\ell}=\sum_{n=1}^{\infty}b_n\cos\frac{n\pi x}{\ell}$$

$$\sum_{n=1}^{\infty}C_n\left[\left(\frac{n\pi}{\ell}\right)^2\sin\frac{n\pi x}{\ell}+\left(\frac{n\pi}{\ell}\right)\cos\frac{n\pi x}{\ell}\right]=\sum_{n=1}^{\infty}-b_n\sin\frac{n\pi x}{\ell}$$

$$C_n=\frac{-b_n\cos\frac{n\pi x}{\ell}}{\left(\frac{n\pi}{\ell}\right)^2\sin\frac{n\pi x}{\ell}+\left(\frac{n\pi}{\ell}\right)\cos\frac{n\pi x}{\ell}}$$

$$where \qquad \ell=2 \ \Rightarrow \qquad C_n=\frac{-b_n\cos\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2\sin\frac{n\pi x}{2}+\left(\frac{n\pi}{2}\right)\cos\frac{n\pi x}{2}}$$

$$\boxed{C_n=\left(\frac{2}{n\pi}\right)(-1)^{\frac{n}{2}}b_n} \qquad n=2,4,6,8,\ldots..n \ \ even$$

$$b_n=\frac{2}{\ell}\int_0^{\ell}f(x)\sin\frac{n\pi x}{\ell}dx=\frac{2}{2}\int_0^{2}x\sin\frac{n\pi x}{2}dx=\left(\frac{4}{n\pi}\right)(-1)^{\frac{n}{2}} \quad n=2,4,6,8,\ldots..n \ \ even$$

$$\boxed{C_n=\frac{8}{(n\pi)^2}(-1)^n} \qquad n=2,4,6,8,\ldots..n \ \ even$$

$$y(x)=\sum_{n=2}^{\infty}C_n\sin n\pi x=\boxed{\frac{8}{\pi^2}\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2}\sin n\pi x}$$

Example 33 : Use Half – Range sine series to solve the differential equation

$$\frac{d^2y}{dx^2}+2y=5\,\delta\left(x-\frac{1}{4}\right) \qquad y=0\ at \qquad x=0,1$$

Solution :

$$\frac{d^2y}{dx^2}+2y=q(x)\ ------(1) \qquad q(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{\ell}\ ------(2)$$

$$y(x)=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi x}{\ell}\ ------(3) \qquad \frac{d^2y}{dx^2}=-\sum_{n=1}^{\infty}C_n\left(\frac{n\pi}{\ell}\right)^2\sin\frac{n\pi x}{\ell}\ ------(4)$$

Sub 2,3 & 4 into 1

$$-\sum_{n=1}^{\infty}C_n\left(\frac{n\pi}{\ell}\right)^2\sin\frac{n\pi x}{\ell}+2\sum_{n=1}^{\infty}C_n\sin\frac{n\pi x}{\ell}=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{\ell}$$

$$-C_n\left(\frac{n\pi}{\ell}\right)^2\cancel{\sum_{n=1}^{\infty}\sin\frac{n\pi x}{\ell}}+2C_n\cancel{\sum_{n=1}^{\infty}\sin\frac{n\pi x}{\ell}}=b_n\cancel{\sum_{n=1}^{\infty}\sin\frac{n\pi x}{\ell}}$$

$$C_n\left[2-\left(\frac{n\pi}{\ell}\right)^2\right]=b_n \qquad \Rightarrow \qquad \boxed{C_n=\frac{b_n\ell^2}{2\ell^2-(n\pi)^2}}$$

$$b_n=\frac{2}{\ell}\int_a^{a+e}\frac{P}{e}\sin\frac{n\pi x}{\ell}dx=\frac{2P}{\ell}\left(-\frac{\ell}{n\pi e}\cos\frac{n\pi x}{\ell}\right)\Bigg|_a^{a+e}=-\frac{2P}{n\pi e}\left(\cos\frac{n\pi(a+e)}{\ell}-\cos\frac{n\pi a}{\ell}\right)$$

$$=-\frac{2P}{n\pi e}\left(\cos\frac{n\pi a}{\ell}\cos\frac{n\pi e}{\ell}-\sin\frac{n\pi a}{\ell}\sin\frac{n\pi e}{\ell}-\cos\frac{n\pi a}{\ell}\right)$$

$$When \qquad e\to 0 \qquad \cos\frac{n\pi e}{\ell}=1 \qquad \sin\frac{n\pi e}{\ell}=\frac{n\pi e}{\ell}$$

$$=-\frac{2P}{n\pi e}\left(\cancel{\cos\frac{n\pi a}{\ell}}-\frac{n\pi e}{\ell}\sin\frac{n\pi a}{\ell}-\cancel{\cos\frac{n\pi a}{\ell}}\right)=-\frac{2P}{\cancel{n\pi e}}\left(-\frac{\cancel{n\pi e}}{\ell}\sin\frac{n\pi a}{\ell}\right)=\frac{2P}{\ell}\sin\frac{n\pi a}{\ell}$$

$$y(x)=\sum_{n=1}^{\infty}\frac{2P\ell}{2\ell^2-(n\pi)^2}\sin\frac{n\pi a}{\ell}\sin\frac{n\pi x}{\ell}$$

$$P=5 \qquad a=1/4 \qquad \ell=1$$

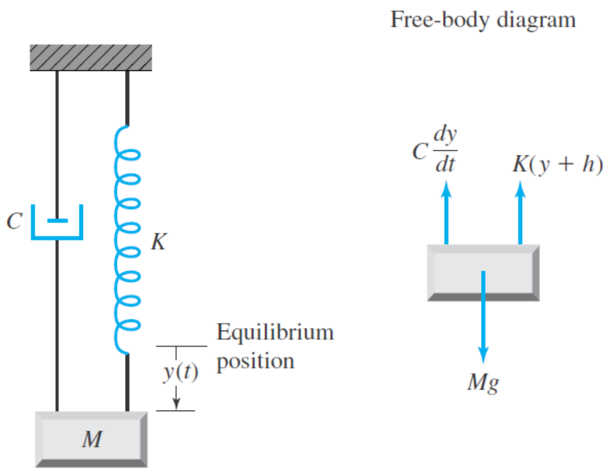
$$y(x)=\sum_{n=1}^{\infty}\frac{2\times5\times1}{2\times1^2-(n\pi)^2}\sin\frac{n\pi}{4}\sin\frac{n\pi x}{1}=\sum_{n=1}^{\infty}\frac{10}{2-(n\pi)^2}\sin\frac{n\pi}{4}\sin n\pi x$$

10.5- Steady – State Force Vibration

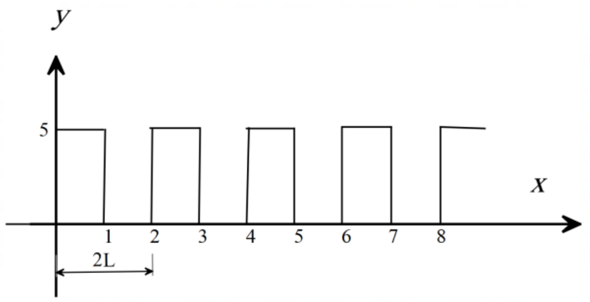
$$M\frac{d^2y}{dx^2}=\sum F \quad \text{in direction of the motion}$$

$$M\frac{d^2y}{dt^2}=-C\frac{dy}{dt}-ky+F(t)$$

$$M\frac{d^2y}{dt^2}+C\frac{dy}{dt}+ky=F(t)$$



Example 33 : Find the Steady-State Force Vibration for M=1 kN.sec^2 /m , C=0.1 , k=50 kN/m.



Solution :

$$M\frac{d^2y}{dt^2}+C\frac{dy}{dt}+ky=F(t)$$

$$\frac{d^2y}{dt^2}+0.1\frac{dy}{dt}+50y=F(t)$$

$$2\ell=2 \quad\Rightarrow\quad \ell=1 \qquad f(t)=5$$

$$a_o=\frac{1}{\ell}\int_0^{2\ell}f(t)dt=\frac{1}{1}\int_0^{1}5dt=5$$

$$a_n=\frac{1}{\ell}\int_0^{2\ell}f(t)\cos\frac{n\pi t}{\ell}dt=\frac{1}{1}\int_0^{2\ell}5\cos(n\pi t)dt=0$$

$$b_n=\frac{1}{\ell}\int_0^{2\ell}f(t)\sin\frac{n\pi t}{\ell}dt=\frac{1}{1}\int_0^{2\ell}5\sin(n\pi t)dt=\frac{10}{n\pi}$$

$$F(t)=\frac{a_o}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi t}{\ell}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi t}{\ell}=\frac{5}{2}+\frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi t)$$

$$\frac{d^2y}{dt^2}+0.1\frac{dy}{dt}+50y=2.5+\frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi t) \qquad ---(1)$$

$$y_P = A + \sum_{n=1}^{\infty} M_n \sin(n\pi t) + \sum_{n=1}^{\infty} K_n \cos(n\pi t)$$

$$-\sum_{n=1}^{\infty} M_n (n\pi)^2 \sin(n\pi t) + \sum_{n=1}^{\infty} K_n (n\pi)^2 \cos(n\pi t)$$

$$+0.1\left\{\sum_{n=1}^{\infty} M_n (n\pi)\cos(n\pi t) - \sum_{n=1}^{\infty} K_n (n\pi)\sin(n\pi t)\right\}$$

$$+50\left\{A + \sum_{n=1}^{\infty} M_n \sin(n\pi t) + \sum_{n=1}^{\infty} K_n \cos(n\pi t)\right\} = 2.5 + \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi t)$$

$$\therefore 50A = 2.5 \qquad A = 0.05$$

$$\left(50-(n\pi)^2\right)M_n - 0.1(n\pi)K_n = \frac{10}{n\pi} \qquad ----(a)$$

$$0.1(n\pi)M_n - \left(50-(n\pi)^2\right)K_n = 0 \qquad ----(b)$$

Solve

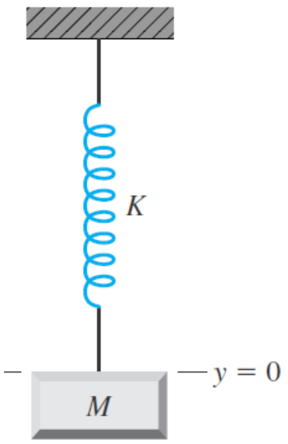
$$M_n = \frac{-1000\left(50-(n\pi)^2\right)}{(n\pi)^3 - 100(n\pi)\left(50-(n\pi)^2\right)^2}$$

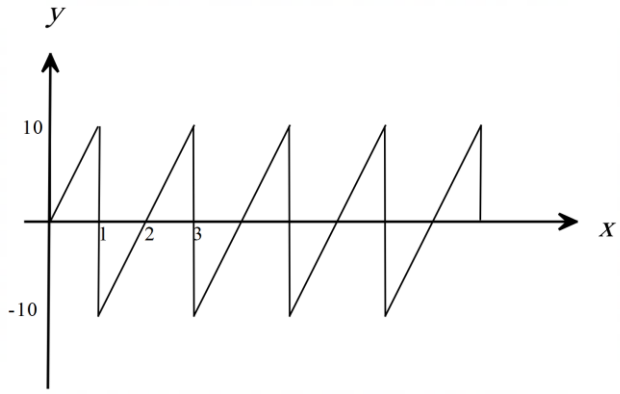
$$K_n = \frac{100}{(n\pi)^2 - 100\left(50-(n\pi)^2\right)^2}$$

sub. in y_P

$$y_P = 0.05 + \sum_{n=1}^{\infty}\frac{-1000\left(50-(n\pi)^2\right)}{(n\pi)^3 - 100(n\pi)\left(50-(n\pi)^2\right)^2}\sin(n\pi t) + \sum_{n=1}^{\infty}\frac{100}{(n\pi)^2 - 100\left(50-(n\pi)^2\right)^2}\cos(n\pi t)$$

H.W : Find the Steady-State Force Vibration for M=1 kN.sec^2 /m , C=0 , k=50 kN/m.





10.6- Free Vibration of Simply Supported Beams

$$\frac{d^4y}{dx^4}=\frac{q(x)}{EI}$$

$$y(x)=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi x}{\ell}$$

$$q(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{\ell} \qquad b_n=\frac{2}{\ell}\int_0^{\ell}q(x)\sin\frac{n\pi x}{\ell}dx$$

$$y(x,0)=y_{Static}$$

$$\because \quad \frac{d^4y}{dx^4}=\frac{q(x)}{EI}$$

$$\because \quad \frac{d^4y}{dx^4}=\sum_{n=1}^{\infty}C_n\left(\frac{n\pi}{\ell}\right)^4\sin\frac{n\pi x}{\ell}$$

$$\therefore \quad \sum_{n=1}^{\infty}C_n\left(\frac{n\pi}{\ell}\right)^4\sin\frac{n\pi x}{\ell}=\frac{\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{\ell}}{EI}$$

$$\sum_{n=1}^{\infty}C_n\left(\frac{n\pi}{\ell}\right)^4\sin\frac{n\pi x}{\ell}=\sum_{n=1}^{\infty}\frac{b_n}{EI}\sin\frac{n\pi x}{\ell}$$

$$C_n\left(\frac{n\pi}{\ell}\right)^4=\frac{b_n}{EI} \qquad \therefore \boxed{C_n=\frac{b_n}{EI}\left(\frac{\ell}{n\pi}\right)^4}$$

$$y_{Static}(x)=\sum_{n=1}^{\infty}\frac{b_n}{EI}\left(\frac{\ell}{n\pi}\right)^4\sin\frac{n\pi x}{\ell}$$

After the load removed

$$q=-\rho A\frac{\partial^2 y}{\partial t^2}$$

$\rho=$ Density perunit Volume

$$\frac{d^4y}{dx^4}=\frac{q(x)}{EI}\qquad\Rightarrow\qquad\frac{\partial^4 y}{\partial x^4}=-\frac{\rho A}{EI}\frac{\partial^2 y}{\partial t^2}$$

$$Let\quad y_{(x,t)}=\sum_{n=1}^{\infty}f_n(t)\sin\frac{n\pi x}{\ell}$$

$Sub.$

$$\sum_{n=1}^{\infty}f_n(t)\left(\frac{n\pi}{\ell}\right)^4\sin\frac{n\pi x}{\ell}=-\frac{\rho A}{EI}\sum_{n=1}^{\infty}f_n''(t)\sin\frac{n\pi x}{\ell}$$

$$\frac{\rho A}{EI}f_n''(t)+\left(\frac{n\pi}{\ell}\right)^4 f_n(t)=0$$

$$\therefore\qquad f_n''(t)+\frac{EI}{\rho A}\left(\frac{n\pi}{\ell}\right)^4 f_n(t)=0$$

$$Let\quad \alpha_n^2=\frac{EI}{\rho A}\left(\frac{n\pi}{\ell}\right)^4\qquad\qquad \alpha_n=\left(\frac{n\pi}{\ell}\right)^2\sqrt{\frac{EI}{\rho A}}$$

$$f_n''(t)+\alpha_n^2 f_n(t)=0\qquad Seound-Order\ DE$$

$$f_n=C_1\cos\alpha_n t+C_2\sin\alpha_n t$$

$$\therefore\qquad y_{(x,t)}=\sum_{n=1}^{\infty}\left(A_n\cos\alpha_n t+B_n\sin\alpha_n t\right)\sin\frac{n\pi x}{\ell}$$

Initial Conditions t=0

$$(1)\qquad \frac{\partial y}{\partial t}(x,0)=0$$

$$(2)\qquad y(x,0)=y_{static}$$

$$\frac{\partial y_{(x,0)}}{\partial t}=\sum_{n=1}^{\infty}\left(-A_n\alpha_n\sin\alpha_n t+B_n\alpha_n\cos\alpha_n t\right)\sin\frac{n\pi x}{\ell}$$

$$0=\sum_{n=1}^{\infty}\left(-A_n\alpha_n(0)+B_n\alpha_n(1)\right)\sin\frac{n\pi x}{\ell}$$

$$B_n\alpha_n=0\qquad where\qquad \alpha_n\neq 0\qquad \therefore B_n=0$$

$$\therefore\qquad y_{(x,t)}=\sum_{n=1}^{\infty}\left(A_n\cos\alpha_n t\right)\sin\frac{n\pi x}{\ell}$$

$$at \qquad t=0 \quad y(x,0)=y_{static}$$

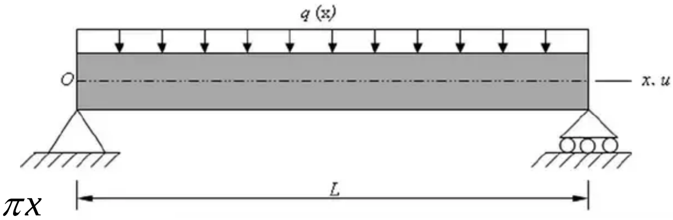
$$\sum_{n=1}^{\infty}\frac{b_n}{EI}\left(\frac{\ell}{n\pi}\right)^4\sin\frac{n\pi x}{\ell}=\sum_{n=1}^{\infty}(A_n(1))\sin\frac{n\pi x}{\ell}$$

$$\therefore \qquad A_n=\frac{b_n}{EI}\left(\frac{\ell}{n\pi}\right)^4$$

$$\therefore \qquad y_{(x,t)}=\sum_{n=1}^{\infty}\left(\frac{b_n}{EI}\left(\frac{\ell}{n\pi}\right)^4\cos\alpha_n t\right)\sin\frac{n\pi x}{\ell}$$

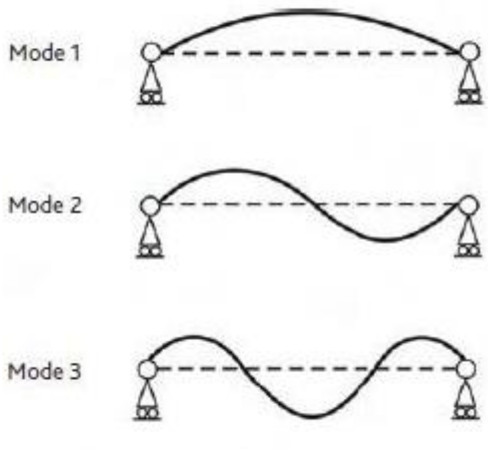
Example 34 : Find the free vibration for simply supported beam with uniform distribution load using Fourier series method.

Solution :



$$b_n=\frac{4q}{n\pi}$$

$$\because \qquad y_{(x,t)}=\sum_{n=1}^{\infty}\left(\frac{4q\ell^4}{EI\,(n\pi)^5}\cos\alpha_n t\right)\sin\frac{n\pi x}{\ell}$$



$$\cos\alpha_n(t+T)=\cos(\alpha_n t+2\pi)$$

$$\alpha_n(t+T)=\alpha_n t+2\pi$$

$$\alpha_n T=2\pi$$

$$T=\frac{2\pi}{\alpha_n}$$

$$\because f=\frac{1}{T}$$

$$f=\frac{1}{2\pi}\alpha_n \qquad where \qquad \alpha_n=\left(\frac{n\pi}{\ell}\right)^2\sqrt{\frac{EI}{\rho A}}$$

$$\therefore f=\frac{1}{2\pi}\left(\frac{n\pi}{\ell}\right)^2\sqrt{\frac{EI}{\rho A}}$$

H.W 1: Find the free vibration for simply supported beam on elastic foundation with uniform distribution load using Fourier series method.

H.W 2: Find the free vibration for simply supported beam with linear distribution load using Fourier series method.

11- Fourier Transforms

11.1- The complex exponential form

$$f(t)=\frac{a_o}{2}+\sum_{n=1}^{\infty}a_n\left(\frac{e^{\frac{ni\pi t}{p}}+e^{-\frac{ni\pi t}{p}}}{2}\right)+\sum_{n=1}^{\infty}b_n\left(\frac{e^{\frac{ni\pi t}{p}}-e^{-\frac{ni\pi t}{p}}}{2i}\right)$$

$$f(t)=\frac{a_o}{2}+\sum_{n=1}^{\infty}\left(\frac{a_n-ib_n}{2}\right)e^{\frac{ni\pi t}{p}}+\sum_{n=1}^{\infty}\left(\frac{a_n+ib_n}{2}\right)e^{-\frac{ni\pi t}{p}}$$

Define

$$C_o=\frac{a_o}{2}\quad,\quad C_n=\frac{a_n-ib_n}{2}\quad,\quad C_{-n}=\frac{a_n+ib_n}{2}$$

$$f(t)=\sum_{-\infty}^{\infty}C_n e^{\frac{ni\pi t}{p}}$$

$$C_o=\frac{a_o}{2}=\frac{1}{2p}\int_d^{d+2p}f(t)dt$$

$$C_n=\frac{a_n-ib_n}{2}=\frac{1}{2p}\left(\int_d^{d+2p}f(t)\cos\left(\frac{n\pi t}{p}\right)dt-i\int_d^{d+2p}f(t)\sin\left(\frac{n\pi t}{p}\right)dt\right)$$

$$=\frac{1}{2p}\left(\int_d^{d+2p}f(t)\left[\cos\left(\frac{n\pi t}{p}\right)-i\sin\left(\frac{n\pi t}{p}\right)\right]dt\right)$$

$$=\frac{1}{2p}\int_d^{d+2p}f(t)e^{-\frac{ni\pi t}{p}}dt$$

$$C_{-n}=\frac{a_n+ib_n}{2}=\frac{1}{2p}\int_d^{d+2p}f(t)e^{\frac{ni\pi t}{p}}dt$$

Where :

$$e^{i\theta}=\cos\theta+i\sin\theta$$

$$e^{-i\theta}=\cos\theta-i\sin\theta$$

$$\therefore\qquad \sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$\therefore\qquad \sin\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}$$

Example 35 : Find the complex form of the Fourier series of $f(t)=e^{-t} \quad -1<t<1$

Solution

$$p=\frac{(+1)-(-1)}{2}=1$$

$$f(t)=\sum_{-\infty}^{\infty} C_n e^{\frac{ni\pi t}{p}}$$

$$C_n=\frac{1}{2p}\int_d^{d+2p} f(t)e^{-\frac{ni\pi t}{p}}dt=\frac{1}{2\times 1}\int_{-1}^{1} e^{-t}e^{-\frac{ni\pi t}{1}}dt=\frac{1}{2}\int_{-1}^{1} e^{-(1+ni\pi)t}dt$$

$$C_n=\frac{-1}{2(1+ni\pi)}\left[e^{-(1+ni\pi)t}\right]_{-1}^{+1}=\frac{e^{(1+ni\pi)}-e^{-(1+ni\pi)}}{2(1+ni\pi)}=\frac{ee^{ni\pi}-e^{-1}e^{-ni\pi}}{2(1+ni\pi)}=\frac{e\left(e^{i\pi}\right)^n-e^{-1}\left(e^{-i\pi}\right)^n}{2(1+ni\pi)}$$

where

$$e^{i\pi}=\cos\pi+i\sin\pi=-1$$

$$e^{-i\pi}=\cos\pi-i\sin\pi=-1$$

$$C_n=\frac{e(-1)^n-e^{-1}(-1)^n}{2(1+ni\pi)}=(-1)^n\frac{e-e^{-1}}{2(1+ni\pi)}=(-1)^n\frac{\sinh 1}{1+ni\pi}$$

$$f(t)=\sinh 1\sum_{-\infty}^{\infty}\frac{(-1)^n}{1+ni\pi}e^{ni\pi t}$$

11.2- The trigonometric form

$$f(t)=\frac{a_o}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi t}{p}\right)+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi t}{p}\right)$$

$$f(t)=\frac{a_o}{2}+\sum_{n=1}^{\infty}\sqrt{a_n^2+b_n^2}\left[\frac{a_n}{\sqrt{a_n^2+b_n^2}}\cos\left(\frac{n\pi t}{p}\right)+\sum_{n=1}^{\infty}\frac{b_n}{\sqrt{a_n^2+b_n^2}}\sin\left(\frac{n\pi t}{p}\right)\right]$$

$$Let\quad A_o=\frac{a_o}{2}\quad ,\quad A_n=\sqrt{a_n^2+b_n^2}$$

$$f(t)=A_o+\sum_{n=1}^{\infty}A_n\left[\cos\left(\frac{n\pi t}{p}\right)\cos\gamma_n+\sum_{n=1}^{\infty}\sin\left(\frac{n\pi t}{p}\right)\sin\gamma_n\right]$$

$$\gamma_n=\frac{n\pi}{p}$$

$$f(t)=A_o+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi t}{p}-\gamma_n\right)$$

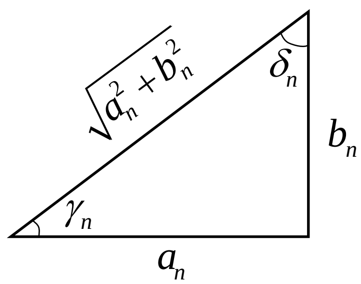
$$f(t)=A_o+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi t}{p}-\frac{n\pi}{p}\right)$$

$$\boxed{f(t)=A_o+\sum_{n=1}^{\infty}A_n\cos\frac{n\pi}{p}(t-1)}$$

OR

$$f(t)=A_o+\sum_{n=1}^{\infty}A_n\left[\cos\left(\frac{n\pi t}{p}\right)\sin\delta_n+\sum_{n=1}^{\infty}\sin\left(\frac{n\pi t}{p}\right)\cos\delta_n\right]$$

$$f(t)=A_o+\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi t}{p}+\delta_n\right)$$



12-The Fourier Integral

Many problems in engineering do not involve periodic functions, and it is therefore non-periodic functions cannot be handled directly by Fourier series. If in a periodic function: $f_p(t)$, we let P approaches infinite, then f(t) is no integer periodic. Begin with complex form:

$$f(t)=\sum_{-\infty}^{\infty} C_n e^{\frac{ni\pi t}{p}} \quad , \qquad C_n=\frac{1}{2p}\int_{-p}^{p} f(t)e^{-\frac{ni\pi t}{p}}dt=\frac{1}{2p}\int_{-p}^{p} f_p(\tau)e^{-\frac{ni\pi\tau}{p}}d\tau$$

$$\therefore f_p(t)=\sum_{-\infty}^{\infty}\left(\frac{1}{2p}\int_{-p}^{p} f_p(\tau)e^{-\frac{ni\pi\tau}{p}}d\tau\right)e^{\frac{ni\pi t}{p}}=\sum_{-\infty}^{\infty}\left[\left(\frac{1}{2\pi}\int_{-p}^{p} f_p(\tau)e^{-\frac{ni\pi\tau}{p}}d\tau\right)e^{\frac{ni\pi t}{p}}\times\frac{\pi}{p}\right]$$

$$Let\ frequency\ \ \omega_n=\frac{n\pi}{p} \quad , \Delta\omega=\frac{\pi}{p}$$

$$f_p(t)=\sum_{-\infty}^{\infty}\left[\left(\frac{1}{2\pi}e^{i\omega_n t}\int_{-p}^{p} f_p(\tau)e^{-i\omega_n t}d\tau\right)\times\Delta\omega\right]$$

$$Let \quad F(\omega)=\frac{e^{i\omega_n t}}{2\pi}\int_{-p}^{p} f_p(\tau)e^{-i\omega_n t}d\tau$$

$$\therefore f_p(t)=\sum_{-\infty}^{\infty} F(\omega_n)\times\Delta\omega \qquad ------(1)$$

$$f(t)=\int_{-\infty}^{\infty}\left[\frac{1}{2\pi}e^{i\omega t}\int_{-\infty}^{\infty} f(\tau)e^{-i\omega t}d\tau\right]d\omega \qquad ------(2)$$

$$f(t)=\int_{-\infty}^{\infty} g(\omega)e^{i\omega t}d\omega \qquad ------(3)$$

$$g(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)e^{-i\omega t}d\tau \qquad ------(4)$$

$$\therefore f(t)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} g(\omega)\cos\omega t\ d\omega$$

$$\therefore g(\omega)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(t)\cos\omega t\, dt$$

$for\ Even\ f(t)$

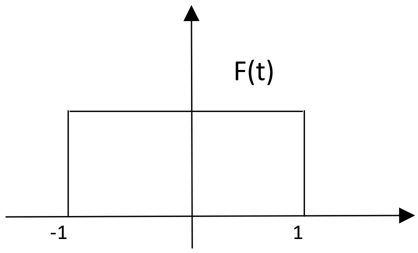
$$\therefore f(t)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} g(\omega)\sin\omega t\ d\omega$$

$$\therefore g(\omega)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(t)\sin\omega t\, dt$$

$for\ Odd\ f(t)$

Example 37 : Find the Fourier integral of

$$f(t)=\begin{cases}1 & |t|<1\\ 0 & |t|>1\end{cases}$$



Solution

$$The\ \ f(t)\ is\ even$$

$$\therefore f(t)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} g(\omega)\cos\omega t\ d\omega$$

$$\therefore g(\omega)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\cos\omega t\, dt$$

$$g(\omega)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}(1)\cos\omega t\, dt=\sqrt{\frac{2}{\pi}}\left(\frac{\sin\omega t}{\omega}\right)\Bigg|_0^1=\sqrt{\frac{2}{\pi}}\frac{\sin\omega}{\omega}$$

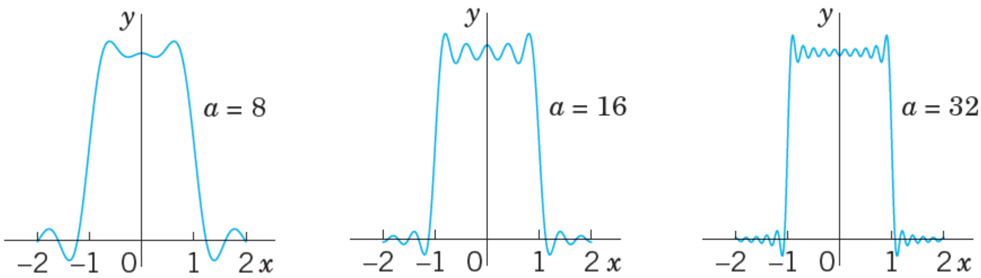
$$f(t)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\sqrt{\frac{2}{\pi}}\frac{\sin\omega}{\omega}\cos\omega t\ d\omega=\frac{2}{\pi}\int_0^{\infty}\frac{\sin\omega}{\omega}\cos\omega t\ d\omega=\frac{2}{\pi}\int_0^{\infty}\frac{1}{\omega}\sin\omega\cos\omega t\ d\omega$$

$$=\frac{2}{\pi}\int_0^{\infty}\frac{1}{\omega}\{\sin(t+1)\omega+\sin(t-1)\omega\}d\omega$$

$$=\frac{2}{\pi}\int_0^{\infty}\frac{\sin(t+1)\omega}{\omega}d\omega+\frac{2}{\pi}\int_0^{\infty}\frac{\sin(t-1)\omega}{\omega}d\omega$$

$$S_i(t)=\int_0^{t}\frac{\sin\omega}{\omega}d\omega \qquad \text{sine integral}$$

$$S_i(\infty)=\int_0^{\infty}\frac{\sin\omega}{\omega}d\omega=\frac{\pi}{2}$$



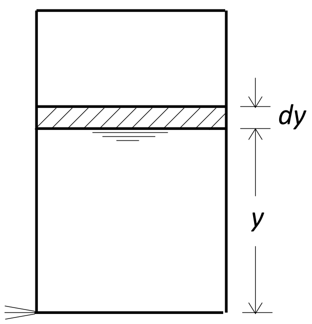
13-Applications on First-Order ODE

The mathematical formulation of physical problems involving continuously changing quantities, often leads to differential equations of the first-order.

13.1- Flow through orifices

Consider a tank which contains any liquid and there is an orifice (hole) at its bottom through which the liquid drains under the influence of gravity. Thus, the depth of water is changed through time. In an interval of time dt, the water level will fall by the amount dy, and the change of volume of liquid inside the tank is equal to the volume of liquid drained outside the tank, i.e,

$$(dV)_{in} = (dV)_{out} \qquad \Rightarrow \qquad A.dy = -Q.dt\,,$$



where,

A is the cross sectional area of the tank.

Q is the discharge of liquid through the orifice $= C_d.a.v$.

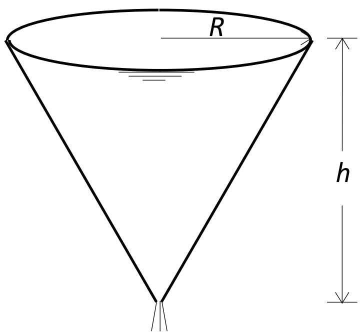
C_d is the coefficient of discharge.

a is the area of the orifice (hole).

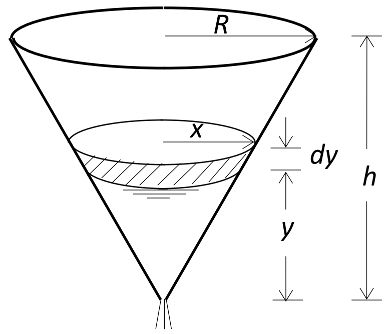
v is the velocity of liquid leaving the orifice $= \sqrt{2gy}$.

$(-ve)$ the negative sign indicates that as *t* increases, *y* decreases.

Example 38 : An inverted right circular conical tank, as shown in the figure, is initially filled with water. The water drains, due to gravity, through a small hole of radius at the bottom. Find the height of water as a function of time and the time required for the tank to drain completely.



Solution:



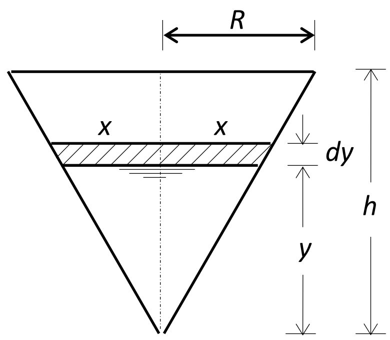
$$(dV)_{in} = (dV)_{out} \quad \Rightarrow \quad A.dy = -Q.dt\,,$$

$$\Rightarrow \quad A.dy = -C_d.a.v.dt\,,$$

$$\Rightarrow \quad \pi x^2.dy = -C_d.\pi r^2.\sqrt{2gy}.dt\,,$$

$$\Rightarrow \quad x^2.dy = -C_d r^2\sqrt{2gy}.dt\,.$$

But $x = \dfrac{Ry}{h} \quad \Rightarrow \quad \dfrac{R^2y^2}{h^2}.dy = -C_d r^2\sqrt{2g}.\sqrt{y}.dt\,.$ (Separable variables DE)



$$\Rightarrow \quad x = \frac{Ry}{h} \quad \frac{x}{y} = \frac{R}{h}$$

$$\therefore \quad \frac{y^2}{\sqrt{y}}.dy = -\frac{C_d r^2 h^2\sqrt{2g}}{R^2}.dt\,,$$

$$\Rightarrow \quad y^{3/2}.dy = -\frac{C_d r^2 h^2\sqrt{2g}}{R^2}.dt\,,$$

$$\Rightarrow \quad \frac{2}{5}y^{5/2} = -\frac{C_d r^2 h^2\sqrt{2g}}{R^2}.t + C\,. \qquad \text{(G.S)}$$

Appling the initial condition (I.C);

Initially, at $t = 0$, the tank is filled with water, $y = h$,

$$\therefore \quad y(0) = h \quad \Rightarrow \quad \frac{2}{5}h^{5/2} = 0 + C \quad \Rightarrow \quad C = \frac{2}{5}h^{5/2}\,.$$

$$\therefore \quad \frac{2}{5}y^{5/2} = -\frac{C_d r^2 h^2\sqrt{2g}}{R^2}.t + \frac{2}{5}h^{5/2} \quad \text{or} \quad y^{5/2} = -\frac{5C_d r^2 h^2\sqrt{2g}}{2R^2}.t + h^{5/2}\,. \qquad \text{(P.S)}$$

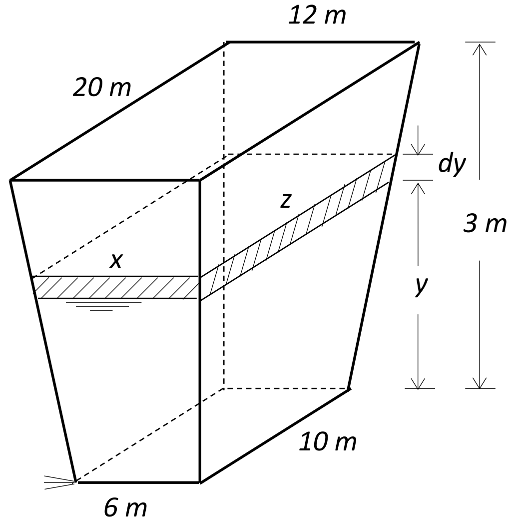
The tank will be empty when $y = 0$,

$$\therefore \quad 0 = -\frac{5C_d r^2 h^2\sqrt{2g}}{2R^2}.t + h^{5/2} \quad \Rightarrow \quad t = \frac{2R^2 h^{5/2}}{5C_d r^2 h^2\sqrt{2g}}\,,$$

$$\text{or} \quad t = \frac{2}{5C_d}.\left(\frac{R}{r}\right)^2.\sqrt{\frac{h}{2g}}\,.$$

Example 39: A water tank, rectangular in cross section, has the dimensions $20\times12m$ at the top and $6\times10m$ at the bottom and is $3m$ in height. It is filled with water and has a circular orifice of $5cm$ diameter at its bottom. Assuming $C_d=0.6$ for the orifice, find the equation of the height of water in the tank with time, then compute the time required for emptying the tank.

Solution:



$$(dV)_{in}=(dV)_{out} \quad\Rightarrow\quad A.dy=-Q.dt\,,$$

$$\Rightarrow \quad A.dy=-C_d.a.v.dt\,,$$

$$\Rightarrow \quad x.z.dy=-C_d.\pi r^2.\sqrt{2gy}.dt\,,$$

$$(2y+6)(\frac{10}{3}y+10)dy=-0.6\pi(\frac{2.5}{100})^2\sqrt{2\times9.81y}.dt\,,$$

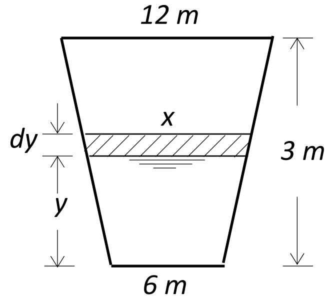
$$\Rightarrow 20(\frac{y^2}{3}+2y+3)dy=-5.218\times10^{-3}\sqrt{y}.dt\,,$$

(Separable variables DE)

$$\therefore \quad (\frac{y^2}{3\sqrt{y}}+\frac{2y}{\sqrt{y}}+\frac{3}{\sqrt{y}})dy=-2.61\times10^{-4}dt\,,$$

$$\Rightarrow \quad (\frac{1}{3}y^{3/2}+2y^{1/2}+3y^{-1/2})dy=-2.61\times10^{-4}dt\,,$$

$$\therefore \frac{2}{15}y^{5/2}+\frac{4}{3}y^{3/2}+6y^{1/2}=-2.61\times10^{-4}.t+C\,. \qquad \text{(G.S)}$$



$$\frac{x-6}{y}=\frac{12-6}{3}$$

$$\therefore x=2y+6$$

Appling the initial condition (I.C);

Initially, at $t=0$, the tank is filled with water, $y=3m$,

$$\therefore \quad y(0)=3 \quad\Rightarrow\quad \frac{2}{15}\times3^{5/2}+\frac{4}{3}\times3^{3/2}+6\times3^{1/2}=0+C\,,$$

$$\Rightarrow \quad C=19.4\,.$$

$$\therefore \frac{2}{15}y^{5/2}+\frac{4}{3}y^{3/2}+6y^{1/2}=-2.61\times10^{-4}.t+19.4\,. \qquad \text{(P.S)}$$

The tank will be empty when $y=0$,

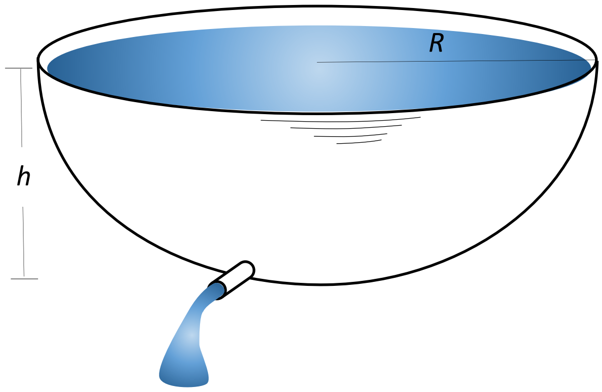
$$\therefore \quad 0=-2.61\times10^{-4}.t+19.4\,,$$

$$\Rightarrow \quad t=74329.5\sec$$

$$\approx 20.65hr$$

Example 39 : Consider a hemisphere tank is fall with water at t=0. Calculate the time of emptying?

Solution:



$$A = \pi x^2$$

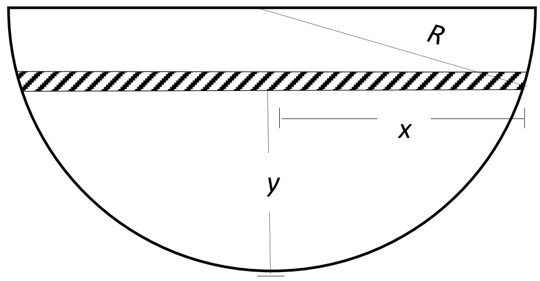
$$(dV)_{in} = A\,dy = \pi x^2 dy$$

$$Q = -C_d\,a v\,dt = -C_d \pi r^2 \sqrt{2gy}\,dt$$

$$(dV)_{out} = -Q dt$$

$$(dV)_{in} = (dV)_{out}$$

$$\pi x^2 dy = -C_d \pi r^2 \sqrt{2gy}\,dt$$



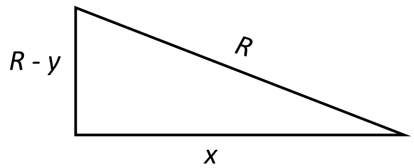
$$\because R^2 = x^2 + (R-y)^2$$

$$x^2 = R^2 - (R-y)^2 = R^2 - R^2 + 2Ry - y^2$$

$$\therefore x^2 = 2Ry - y^2$$

$$(2Ry - y^2)dy = -C_d r^2 \sqrt{2gy}\,dt$$

$$\frac{(2Ry - y^2)}{\sqrt{y}}dy = -C_d r^2 \sqrt{2g}\,dt$$



$$\int \left(2Ry^{\frac{1}{2}} - y^{\frac{3}{2}}\right)dy = -C_d r^2 \sqrt{2g} \int dt + C$$

$$\frac{4}{3}Ry^{\frac{3}{2}} - \frac{2}{5}y^{\frac{5}{2}} = -C_d r^2 \sqrt{2g}\,t + C$$

$$B.C. \qquad at\ t = 0 \qquad\qquad y = R$$

$$\frac{4}{3}R \times R^{\frac{3}{2}} - \frac{2}{5}R^{\frac{5}{2}} = C$$

$$C = \frac{14}{15}R^{\frac{5}{2}}$$

$$\frac{4}{3}Ry^{\frac{3}{2}} - \frac{2}{5}y^{\frac{5}{2}} = -C_d r^2 \sqrt{2g}\,t + \frac{14}{15}R^{\frac{5}{2}}$$

$$\therefore \quad \boxed{t = \frac{\frac{14}{15}R^{\frac{5}{2}} - \frac{4}{3}Ry^{\frac{3}{2}} + \frac{2}{5}y^{\frac{5}{2}}}{C_d r^2 \sqrt{2g}}}$$

The the to empty y = 0

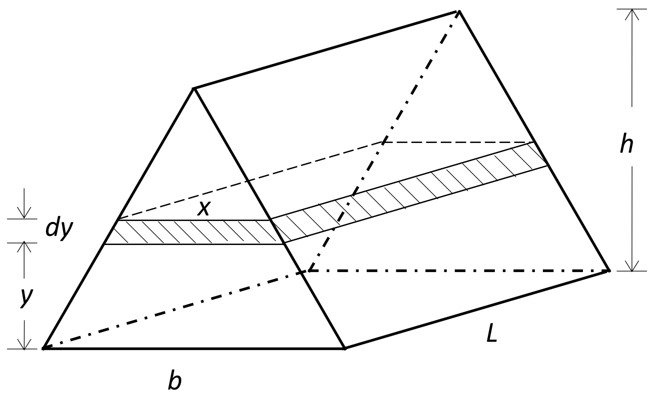
$$\boxed{t_{\max} = \frac{14R^{\frac{5}{2}}}{C_d 15 r^2 \sqrt{2g}}}$$

Useful expressions,

$A = x.L\,.$

$\dfrac{x}{b} = \dfrac{h-y}{h},$

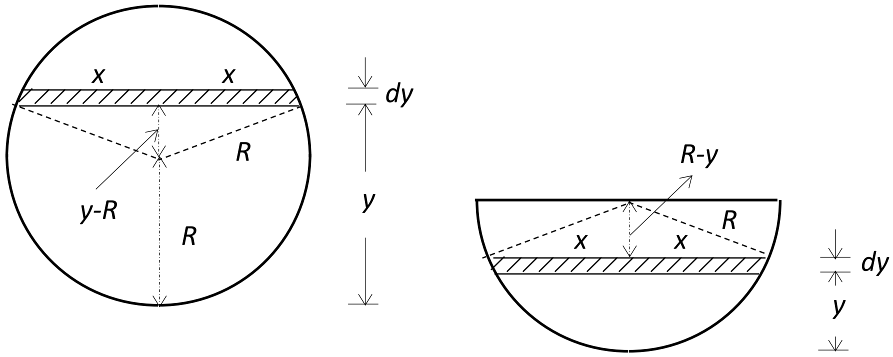
$\therefore\ x = \dfrac{b}{h}.(h-y)\,.$



$A = \pi x^2\,.$

$x^2 + (y-R)^2 = R^2,$

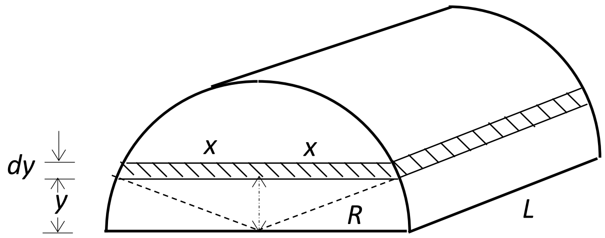
$\therefore\ x^2 = 2Ry - y^2\,.$



$A = 2x.L\,.$

$x^2 + y^2 = R^2,$

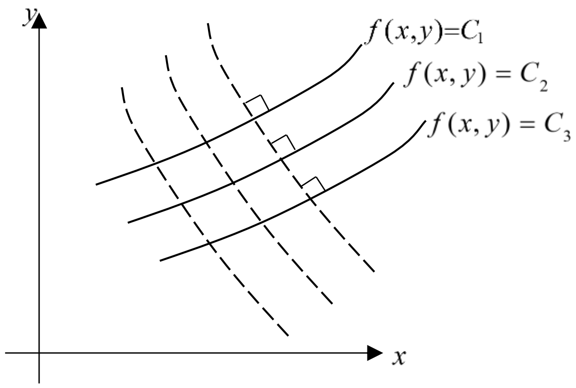
$\therefore\ x = \sqrt{R^2 - y^2}\,.$



13.2- Seepage of soil (Orthogonal Trajectories)

In many engineering problems, a family (set) of curves is given and it is required to find another family whose curves intersect each of the given curves at right angle.

Consider the function $f(x,y)=C$ where *C* is a constant. By changing the value of the constant *C*, a family (set) of curves are obtained for $f(x,y)$, where each curve has one value of the constant. It is required to find another set of curves which are orthogonal to the first set. This is done by eliminating the constant *C* from $f(x,y)=C$ by differentiation, and then replacing $\frac{dy}{dx}$ of these curves by $[-1/\frac{dy}{dx}]$ to get the required orthogonal set.



Example 40: Find the orthogonal trajectories of $xy=C$.

Solution:

Step ***1****;* Find the slope of the given set,

By differentiation $xdy+ydx=0 \quad \Rightarrow \quad \left(\frac{dy}{dx}\right)_1 = \frac{-y}{x}$.

Step ***2****;* Find the slope of the required set,

$$\left(\frac{dy}{dx}\right)_2 = -1/\left(\frac{dy}{dx}\right)_1 \Rightarrow \left(\frac{dy}{dx}\right)_2 = -1/\left(\frac{-y}{x}\right) \Rightarrow \left(\frac{dy}{dx}\right)_2 = \frac{x}{y}.$$

Step ***3****;* Find the required set,

$$\frac{dy}{dx}=\frac{x}{y} \quad \text{(separable variables DE)} \quad \Rightarrow \quad ydy = x.dx,$$

$$\therefore \frac{y^2}{2}=\frac{x^2}{2}+K_1 \quad \text{or} \quad y^2-x^2=K. \quad [K=2K_1]$$

Example 41 : Find the orthogonal trajectories of $y = Cx^2$.

Solution:

*Step **1**;* Find the slope of the given set,

Method I,

$y = Cx^2 \quad \Rightarrow \quad \dfrac{y}{x^2} = C$. By differentiation $\dfrac{x^2.dy - y.(2xdx)}{x^4} = 0$,

$\Rightarrow \quad x^2dy - 2xydx = 0 \quad \Rightarrow \quad \left(\dfrac{dy}{dx}\right)_1 = \dfrac{2y}{x}$. (The slope of the given set)

Method II,

$y = Cx^2$. By differentiation $dy = 2Cxdx \quad \Rightarrow \quad \left(\dfrac{dy}{dx}\right)_1 = 2Cx$.

From the given set $C = \dfrac{y}{x^2} \quad \Rightarrow \quad \therefore \left(\dfrac{dy}{dx}\right)_1 = 2\left(\dfrac{y}{x^2}\right)x \quad \Rightarrow \quad \left(\dfrac{dy}{dx}\right)_1 = \dfrac{2y}{x}$.

*Step **2**;* Find the slope of the required set, Since the required and given sets are orthogonal, then $\left(\dfrac{dy}{dx}\right)_2 = -1/\left(\dfrac{dy}{dx}\right)_1$.

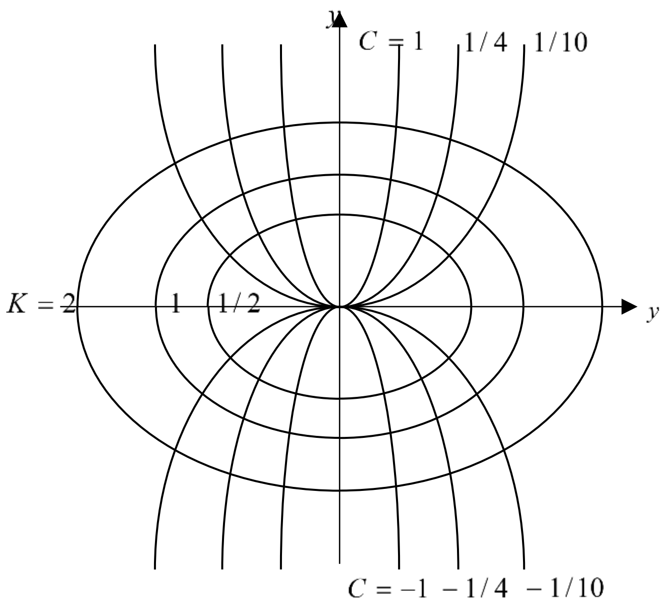
$\therefore \left(\dfrac{dy}{dx}\right)_2 = -1/\left(\dfrac{2y}{x}\right) \quad \Rightarrow \quad \left(\dfrac{dy}{dx}\right)_2 = \dfrac{-x}{2y}$.

*Step **3**;* Find the required set, To find the required set we must solve the above differential equation,

$\dfrac{dy}{dx} = \dfrac{-x}{2y}$ (separable variables DE)

$\Rightarrow \quad 2ydy = -x.dx$,

$\therefore \; y^2 = \dfrac{-x^2}{2} + K \quad$ or $\quad y^2 + \dfrac{x^2}{2} = K$.



Notes,

* *K must* be positive since it is the sum of two squares.

* $y = Cx^2$ is a family of parabolas.

* $y^2 + \dfrac{x^2}{2} = K$ is a family of ellipses.

Example 41 : Find the orthogonal trajectories of $y^2 + x^2 = r^2$.

Solution:

$$y^2 + x^2 = r^2$$

$$2y\frac{dy}{dx} + 2x = 0$$

$$\left(\frac{dy}{dx}\right)_1 = -\frac{x}{y}$$

$$\left(\frac{dy}{dx}\right)_2 = \frac{-1}{\left(\frac{dy}{dx}\right)_1}$$

$$\left(\frac{dy}{dx}\right)_2 = \frac{-1}{-\frac{x}{y}} = \frac{y}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y}{x}$$

$$\frac{dy}{y} = \frac{dx}{x}$$

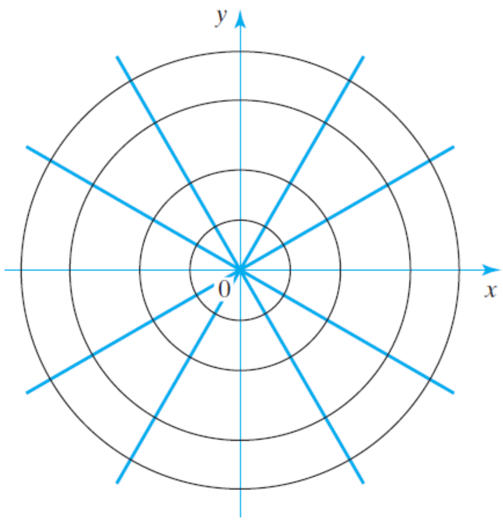
$$\int \frac{dy}{y} = \int \frac{dx}{x} + C_1$$

$$\ln y = \ln x + C_1$$

$$\ln y = \ln x + \ln C_2$$

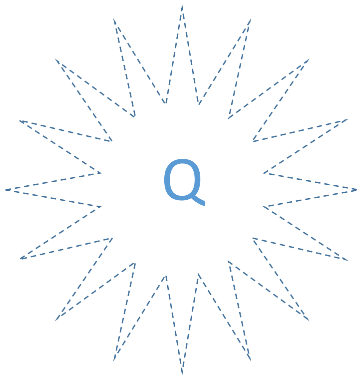
$$\ln y = \ln |C_2 x|$$

$$\boxed{\therefore y = Cx}$$



13.3- Radiation and disintegration of radio active substance

The disintegration rate by radiation is directly proportional to the radio active substance present at any time t.



$$-\frac{dQ}{dt} \propto Q \quad \text{Similar to the rate of flow}$$

$$-\frac{dQ}{dt} = kQ$$

$$\frac{dQ}{Q} = -k\,dt$$

$$\int \frac{dQ}{Q} = \int -k\,dt + C$$

$$\ln Q = -kt + C$$

$$Q = e^{-kt+C}$$

$$Q = e^{C} e^{-kt} \qquad A = e^{C}$$

$$Q = A e^{-kt}$$

$$B.C. \qquad at \;\; t = 0 \qquad Q = Q_o$$

$$Q_o = A e^{-k(0)} \qquad A = Q_o$$

$$\therefore \boxed{Q = Q_o e^{-kt}}$$

Example: The disintegration rate by radiation. If Q_o=0.6 kg at t=0 and Q=0.3 kg at t=550 years find k?

Solution :

$$Q = Q_o e^{-kt}$$

$$0.3 = 0.6 e^{-k(550)}$$

$$\frac{1}{2} = e^{-550k}$$

$$\ln\frac{1}{2} = \ln\left(e^{-550k}\right)$$

$$-\ln 2 = -550k$$

$$k = \frac{\ln 2}{550} = 0.00126$$

$$Q = 0.6 e^{-0.00126t}$$

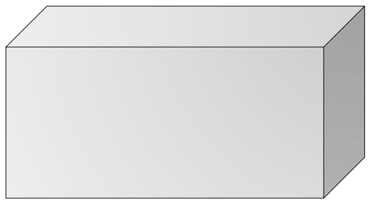
13.4- Heat Transfer Analysis

The three modes of heat Transmission

- Heat Conduction is solids.
- Heat Convection in fluids.
- Radiation of Heat in space.

Derivation of Fourier Law of heat conduction on a solid slab. With the left surface maintained at temperature T_a and the right surface at T_b, Heat will flow from the left to right surface if $T_a > T_b$. By observations, we can formulate the total amount of heat flow (Q) through the thickness of the slab as :

$$Q \propto \frac{A(T_a - T_b)t}{d} \qquad \Rightarrow \qquad Q = k\frac{A(T_a - T_b)t}{d}$$



Where : A : Area m^2
T : time allowing heat flow sec
D : the distance of heat flow m
K : Thermal conductivity (W/m.k)

$$q = \frac{Q}{At} \qquad \textit{Heat flux}$$

$$q = \frac{Q}{At} = k\frac{A(T_a - T_b)t}{d} \qquad (w/m^2)$$

$$q = k\frac{T(x) - T(x+\Delta x)}{\Delta x} = -k\frac{T(x+\Delta x) - T(x)}{\Delta x}$$

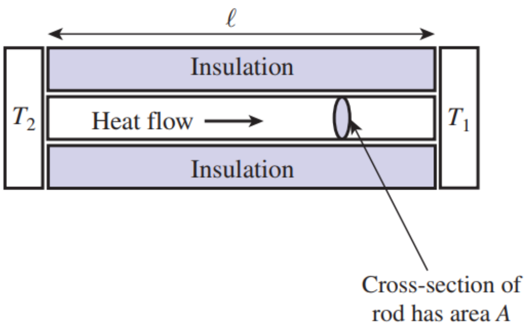
$$q(x) = \lim_{\Delta x \to 0}\left(-k\frac{T(x+\Delta x) - T(x)}{\Delta x}\right) = -k\frac{dT(x)}{dx}$$

$$\boxed{q(x) = -k\frac{dT(x)}{dx}} \qquad \textit{Fourier Law of heat conduction}$$

Material	k
Wood	0.087
Cork	0.039
Air	0.026
Water	0.6
Glass wool	0.04
Rock wool	0.045
steel	50
Fiberglass	0.04
vacuum	0

Example : A metal rod has a cross-sectional area 1200mm^2 and 2 m length. It is thermally insulated in its circumference, with one end being in constant with heat source supplying heat at 10 kW and the other end maintained at 50°C. Determine the temperature distribution in the rod, if the thermal conductivities of the rod material is 100kW/m°c.

Solution



The total heat flow Q per unit time t (Q/t)

$$q = \frac{Q}{At}$$

$$\frac{Q}{t} = Aq = 10kW \quad (let\ t = 1\ unit)$$

$$q(x) = -k\frac{dT(x)}{dx}$$

$$Q = qA = -k\frac{dT(x)}{dx}$$

$$\frac{dT(x)}{dx} = -\frac{Q}{kA} = -\frac{10}{100\times 1200\times 10^{-6}}$$

$$\frac{dT(x)}{dx} = -\frac{250}{3}$$

1^{st} *Order D.E*

$$dT(x) = -\frac{250}{3}dx$$

$$\int dT(x) = \int -\frac{250}{3}dx + C$$

$$T(x) = -\frac{250}{3}x + C$$

$$B.C.\quad at\ x = 2m \qquad T(2) = 50^oC$$

$$50 = -\frac{250}{3}(2) + C$$

$$C = \frac{650}{3}$$

$$\boxed{T(x) = \frac{50}{3}(13 - 5x)}$$

14- Applications on Second and Higher Order Linear

14.1- Buckling of columns

Example 1 : Determine the critical buckling load of a hinged-hinged column.

Solution:

Consider a column of length L , as shown in Fig.(a) or Fig.(b), hinged at both ends, and subjected to a compressive axial force P.

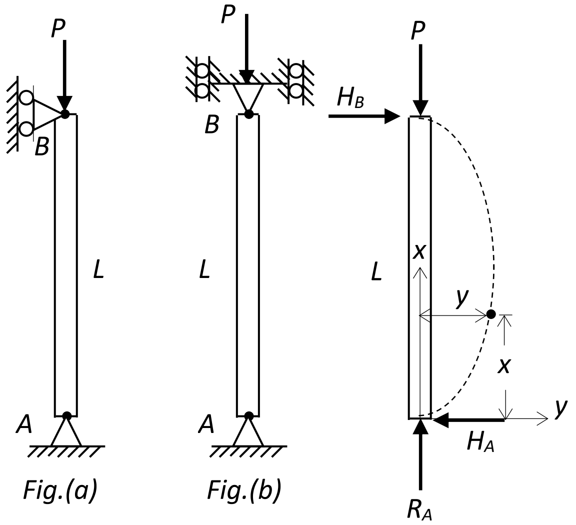


Fig.(a) Fig.(b)

$$EI.\frac{d^2y}{dx^2}=-M_x. \qquad \text{But} \quad M_x=P.y,$$

$$\therefore\ EI.\frac{d^2y}{dx^2}=-P.y \quad \Rightarrow \quad \frac{d^2y}{dx^2}+\frac{P}{EI}y=0.$$

Let $\beta^2=\frac{P}{EI}$ $\Rightarrow$ $\frac{d^2y}{dx^2}+\beta^2y=0$,

or $(D^2+\beta^2)y=0$ $\Rightarrow$ $m^2+\beta^2=0$,

$\Rightarrow$ $m^2=-\beta^2$ $\Rightarrow$ $m_{1,2}=\pm\beta i$,

$\therefore$ $y=C_1\cos\beta x+C_2\sin\beta x$. (G.S)

Boundary conditions,

1. $y(0)=0$ $\Rightarrow$ $0=C_1+0$ $\Rightarrow$ $C_1=0$.

   $\therefore$ $y=C_2\sin\beta x$.

2. $y(L)=0$ $\Rightarrow$ $0=C_2\sin\beta L$.

   If $C_2=0$ $\Rightarrow$ $y=0$.

   (i.e. the column remains straight)

$\therefore$ $C_2\neq0$ $\Rightarrow$ $\sin\beta L=0$ $\Rightarrow$ $\beta L=0,\pi,2\pi,.......,n\pi$,

$\therefore$ $\beta L=n\pi$ $\Rightarrow$ $\beta=\frac{n\pi}{L}$. $(n=1,2,3,.....)$

To determine M_x :

Either from down (left);

$M_x=R_A.y+H_A.x$.

$\sum F_x=0$,

$R_A-P=0$ $\Rightarrow$ $R_A=P$.

$\sum F_y=0$,

$H_B-H_A=0$ $\Rightarrow$ $H_A=H_B$.

$\sum(M)_B=0$,

$H_A.L=0\Rightarrow H_A=0$,

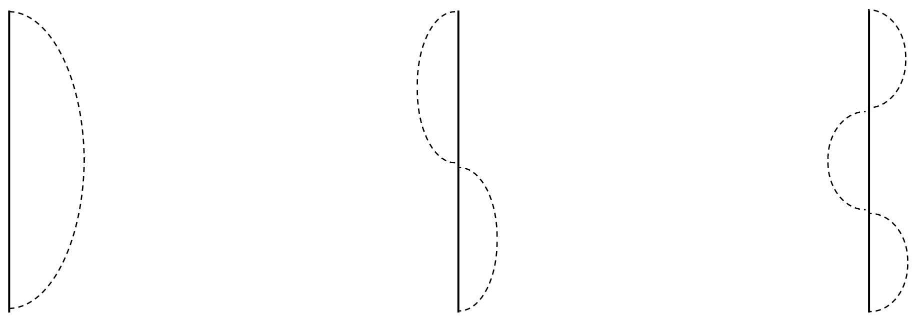
$\therefore H_B=0$.

$\therefore M_x=P.y$

But $\beta^2=\frac{P}{EI}$ $\Rightarrow$ $\frac{P}{EI}=\frac{n^2\pi^2}{L^2}$ $\Rightarrow$ $P=\frac{n^2\pi^2EI}{L^2}$.

For $n=1$ $\Rightarrow$ $P_{cr}=\frac{\pi^2EL}{L^2}$, (Euler load or critical load)

and $y=C_2\sin\frac{\pi x}{L}$. (C_2 remains indeterminate, that is $y(\frac{L}{2})=C_2$)



$n=1$ $n=2$ $n=3$

$$P_{cr}=\frac{\pi^2 EL}{L^2} \qquad P_{cr}=\frac{4\pi^2 EL}{L^2} \qquad P_{cr}=\frac{9\pi^2 EL}{L^2}$$

Critical buckling load:

is the smallest value of axial load that can cause buckling:

* If $P<P_{cr}$, then the case is “*stable equilibrium*”. In this case, no buckling would occur. If lateral deflection is produced, by a horizontal force, then this deflection vanishes when the horizontal force is removed.

* If $P=P_{cr}$, then the case is “*neutral equilibrium*”. In this case, small and limited buckling may occur. If lateral deflection is produced, by a horizontal force, then this deflection remains constant even when the horizontal force is removed.

* If $P>P_{cr}$, then the case is “*unstable equilibrium*”. In this case, large not-controlled buckling may occur. If lateral deflection is produced, by a horizontal force, then this deflection will be increased, and if not controlled, the column will collapse.

Example 2: Determine the critical buckling load of a fixed-fixed ends column.

At each ends there are bending moment and shear force

$$\sum M_A = 0 \qquad VL + M_o - M_o = 0$$

$$VL = 0 \qquad L \neq 0 \qquad \therefore V = 0$$

Bending moment at distance x

$$M = Py - M_o$$

$$\frac{d^2y}{dx^2} = -\frac{M}{EI}$$

$$\frac{d^2y}{dx^2} = -\frac{(Py - M_o)}{EI}$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{M_o}{EI}$$

2nd Order – Nonhomogeneous D.E with constant coefficients

$$\frac{d^2y}{dx^2} + k^2 y = 0 \qquad let \qquad k^2 = \frac{P}{EI}$$

$$y_h = C_1 \cos kx + C_2 \sin kx$$

$$Let \qquad y_p = A$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{M_o}{EI}$$

$$y_p' = y_p'' = 0$$

$$0 + \frac{P}{EI}A = \frac{M_o}{EI} \qquad \therefore A = \frac{M_o}{P}$$

$$y = y_h + y_p$$

$$y = C_1 \cos kx + C_2 \sin kx + \frac{M_o}{P}$$

$$B.C.$$

$$y = y' = 0 \qquad at \;\; x = 0$$

$$0 = C_1 \cos(0) + C_2 \sin(0) + \frac{M_o}{P} \qquad \Rightarrow \boxed{C_1 = -\frac{M_o}{P}}$$

$$y' = -C_1 k \sin kx + C_2 k \cos kx$$

$$0 = -C_1 k \sin(0) + C_2 k \cos(0) \qquad \Rightarrow \boxed{C_2 = 0}$$

$$y = -\frac{M_o}{P}\cos kx + \frac{M_o}{P}$$

$$y = \frac{M_o}{P}(1-\cos kx)$$

$$y = y' = 0 \qquad at\ x = L$$

$$0 = \frac{M_o}{P}(1-\cos kL) \qquad \frac{M_o}{P} \neq 0$$

$$0 = 1-\cos kL \qquad kL = \cos^{-1} 1$$

$$kL = 2n\pi \qquad k^2 = \left(\frac{2n\pi}{L}\right)^2$$

$$\frac{P}{EI} = \left(\frac{2n\pi}{L}\right)^2 \qquad P = \left(\frac{2n\pi}{L}\right)^2 EI \qquad n = 1,2,3,4$$

$$P_{Cr} = \frac{\pi^2 EI}{\left(\frac{L}{2}\right)^2}$$

Example 3: Determine the critical buckling load of a fixed-hinged ends column.

Bending moment at distance x

$$M = Py - H(L-x)$$

$$\frac{d^2y}{dx^2} = -\frac{M}{EI}$$

$$\frac{d^2y}{dx^2} = -\frac{\left(Py - H(L-x)\right)}{EI}$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{H(L-x)}{EI}$$

2nd Order – Nonhomogeneous D.E with constant coefficients

$$\frac{d^2y}{dx^2} + k^2y = 0 \qquad let \qquad k^2 = \frac{P}{EI}$$

$$y_h = C_1\cos kx + C_2\sin kx$$

$$Let \qquad y_p = A + Bx$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{HL}{EI} - \frac{H}{EI}x$$

$$y_p' = B \qquad\qquad y_p'' = 0$$

$$0 + \frac{P}{EI}(A + Bx) = \frac{HL}{EI} - \frac{H}{EI}x$$

$$PA + PBx = HL - Hx \qquad\qquad \therefore A = \frac{HL}{P} \quad , \qquad B = -\frac{H}{P}$$

$$y = y_h + y_p$$

$$y = C_1\cos kx + C_2\sin kx + \frac{H}{P}(L-x)$$

$$B.C.$$

$$y = 0 \qquad at \;\; x = 0$$

$$0 = C_1\cos(0) + C_2\sin(0) + \frac{H}{P}(L-0) \qquad \Rightarrow \boxed{C_1 = -\frac{HL}{P}}$$

$$y' = 0 \qquad at \;\; x = 0$$

$$y' = -C_1k\sin kx + C_2k\cos kx - \frac{H}{P}$$

$$0 = -C_1k\sin(0) + C_2k\cos(0) \qquad \Rightarrow \boxed{C_2 = \frac{H}{Pk}}$$

$$y = -\frac{HL}{P}\cos kx + \frac{H}{Pk}\sin kx + \frac{H}{P}(L - x)$$

$$y = \frac{H}{P}\left(\frac{1}{k}\sin kx - L\cos kx + L - x\right)$$

$$y = 0 \qquad at \ x = L$$

$$0 = \frac{H}{P}\left(\frac{1}{k}\sin kL - L\cos kL + L - L\right)$$

$$0 = \frac{H}{P}\left(\frac{1}{k}\sin kL - L\cos kL\right) \qquad \frac{H}{P} \neq 0$$

$$\frac{1}{k}\sin kL - L\cos kL = 0 \qquad \frac{1}{k}\sin kL = L\cos kL$$

$$\tan(kL) = kL \qquad kL = 4.49341n \qquad n = 1,2,3,4,\ldots(solution\,of\ equation)$$

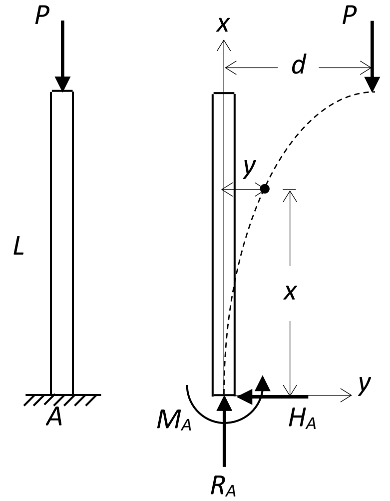
$$k = \frac{4.49341}{L} \qquad n = 1$$

$$k^2 = \frac{20.19}{L^2}$$

$$\boxed{P_{Cr} = \frac{20.19EI}{L^2}}$$

Example 4: Determine the critical buckling load of a fixed-free column.

Solution:



Let the buckling at the free end is (*d*).

$EI.\dfrac{d^2y}{dx^2} = -M_x$. But $M_x = -P(d-y)$,

$\therefore EI.\dfrac{d^2y}{dx^2} = -[-P(d-y)] \Rightarrow \dfrac{d^2y}{dx^2} + \dfrac{P}{EI}y = \dfrac{P}{EI}d$.

Let $\beta^2 = \dfrac{P}{EI} \Rightarrow \dfrac{d^2y}{dx^2} + \beta^2 y = \beta^2 d$,

or $(D^2+\beta^2)y = \beta^2 d \Rightarrow m^2 + \beta^2 = 0$,

$\Rightarrow m^2 = -\beta^2 \Rightarrow m_{1,2} = \pm\beta i$,

$\therefore y_c = C_1 \cos\beta x + C_2 \sin\beta x$.

Let $y_p = A \Rightarrow y'_p = y''_p = 0$.

Substituting,

$0 + \beta^2 A = \beta^2 d \Rightarrow A = d \Rightarrow y_p = d$.

$y = y_c + y_p$,

$\therefore y = C_1 \cos\beta x + C_2 \sin\beta x + d$. (G.S)

To determine M_x:

Either from up (right);

$M_x = -P(d-y)$.

Or from down (left);

$M_x = R_A.y + H_A.x - M_A$.

$\sum F_x = 0$,

$R_A - P = 0 \Rightarrow R_A = P$.

$\sum F_y = 0$,

$-H_A = 0 \Rightarrow H_A = 0$.

Boundary conditions,

1. $y(0) = 0 \Rightarrow 0 = C_1 + d \Rightarrow C_1 = -d$.

   $\therefore y = -d\cos\beta x + C_2 \sin\beta x + d$.

2. $y'(0) = 0$, $y' = -\beta C_1 \sin\beta x + \beta C_2 \cos\beta x \Rightarrow 0 = 0 + \beta C_2 \Rightarrow C_2 = 0$.

   $\therefore y = -d\cos\beta x + d$ or $y = d(1-\cos\beta x)$.

3. $y(L) = d \Rightarrow d = d(1-\cos\beta L) \Rightarrow 1 = 1 - \cos\beta L$,

   $\Rightarrow \cos\beta L = 0 \Rightarrow \beta L = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \ldots\ldots, \dfrac{(2n-1)\pi}{2}$,

   $\therefore \beta L = \dfrac{(2n-1)\pi}{2} \Rightarrow \beta = \dfrac{(2n-1)\pi}{2L}$. $(n = 1,2,3,\ldots..)$

But $\beta^2 = \dfrac{P}{EI} \Rightarrow \dfrac{P}{EI} = \dfrac{(2n-1)^2\pi^2}{4L^2} \Rightarrow P_{cr} = \dfrac{(2n-1)^2\pi^2 EL}{4L^2}$.

For $\Rightarrow P_{cr} = \dfrac{\pi^2 EL}{4L^2}$. (i.e. $\dfrac{1}{4}$ times the critical load for hinged-hinged case)

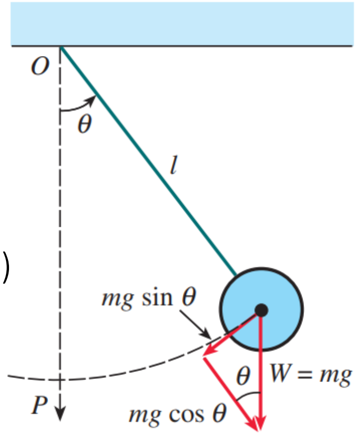
14.2- The pendulum

Example : Study the simple harmonic motion of pendulum without air resistance

Solution

At any time, let the pendulum make an angle θ (radian) with vertical. there arc is S:

$mg\sin\theta$ moving force



Along arc

*Mass*acceleration = Force*

$$m\times\left(-\frac{d^2s}{dt^2}\right)=m\,g\sin\theta \qquad \Rightarrow \qquad \frac{d^2s}{dt^2}+g\sin\theta=0$$

The negative sign means S is decreasing with time (central force)

$$S=\theta L \qquad \frac{ds}{dt}=L\frac{d\theta}{dt} \qquad \frac{d^2s}{dt^2}=L\frac{d^2\theta}{dt^2}$$

$$\frac{d^2s}{dt^2}+g\sin\theta=0 \qquad L\frac{d^2\theta}{dt^2}+g\sin\theta=0$$

$$\frac{d^2\theta}{dt^2}+\frac{g}{L}\sin\theta=0$$

For small values of $\theta=\sin\theta\simeq 0$

$$\frac{d^2\theta}{dt^2}+\frac{g}{L}\theta=0 \qquad k^2=\frac{g}{L}$$

$$\theta=C_1\cos kt+C_2\sin kt$$

$B.C$

$$t=0 \qquad \theta=\theta_o \qquad \theta'=0\ Velocity$$

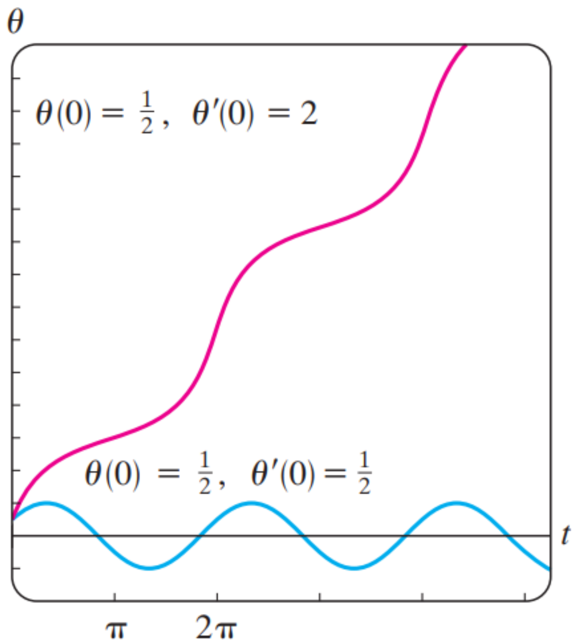
$$\theta_o=C_1\cos(0)+C_2\sin(0) \qquad C_1=\theta_o$$

$$\theta'=-C_1k\sin kt+C_2k\cos kt$$

$$0=-C_1k\sin(0)+C_2k\cos(0) \qquad C_2=0$$

$$\theta=\theta_o\cos kt$$

$$\boxed{\theta=\theta_o\cos\sqrt{\frac{g}{L}}t}$$



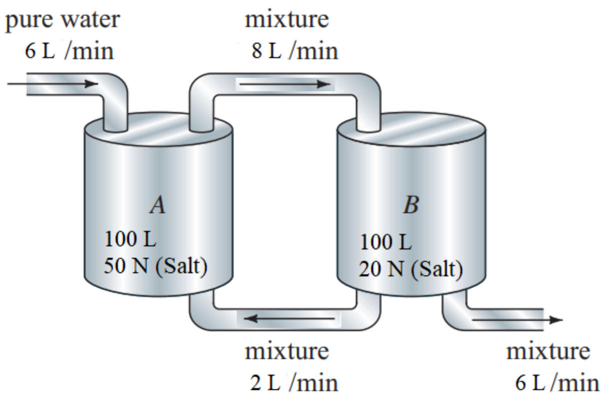
$$\cos\left(\sqrt{\frac{g}{L}}t+2\pi\right)=\cos\left(\sqrt{\frac{g}{L}}(t+T)\right)$$

$$\sqrt{\frac{g}{L}}T=2\pi \qquad T=2\pi\sqrt{\frac{L}{g}}$$

$$f=\frac{1}{T} \qquad f=\frac{1}{2\pi}\sqrt{\frac{g}{L}} \qquad \text{frequency}(\text{Hiz=cycles/sec})$$

14.3- Mixing Problem Involving Two Tanks

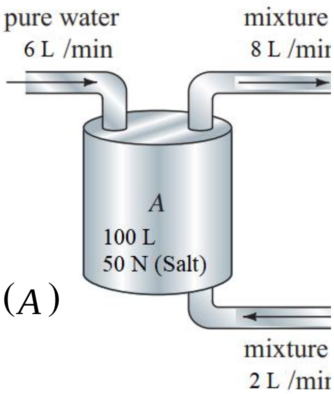
Example : Two tank connected as shown below, first tank contains initially (100 L) of brine containing (50 N) salt , while the second tank contains (100 L) of brine with (20 N) salt is dissolved. Starting (t=0), the pumping was applied. If the brine at each tank I kept uniform by staining. Find the amount of salt in each tank at any time?

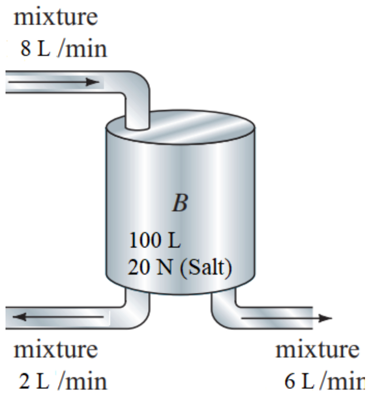


Solution :

x = Weight of salt in tank (A)
y = Weight of salt in tank (B)
(x/100)= Concentration of salt in tank (A)
(y/100)= Concentration of salt in tank (B)

Tank (A)





$$\left(\frac{N}{\ell}\cdot\frac{\ell}{\min}\right)dt = dx$$

$$\left(\frac{2y}{100}-\frac{8x}{100}\right)dt = dx$$

$$0.02y - 0.08x = \frac{dx}{dt} \qquad --(A)$$

Tank (B)

$$\left(\frac{-2y}{100}+\frac{8x}{100}-\frac{6y}{100}\right)dt = dy$$

$$0.08x - 0.08y = \frac{dy}{dt} \qquad --(B)$$

— — — — — — — — — — — —

$$(D+0.08)x - 0.02y = 0 \qquad --(1)$$

$$(0.08)x - (D+0.08)y = 0 \qquad --(2)$$

$$\begin{bmatrix} D+0.08 & -0.02 \\ 0.08 & -(D+0.08) \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left\{-(D+0.08)^2 + 0.08\times 0.02\right\}(x, y) = 0$$

$$-(r+0.08)^2+0.08\times 0.02=0$$

$$r^2+0.16r+0.0048=0$$

$$r_1=-0.04 \quad \& \quad r_2=0.12$$

$$\therefore \quad x=C_1e^{-0.04t}+C_2e^{-0.12t}$$

$$\therefore \quad y=C_3e^{-0.04t}+C_4e^{-0.12t}$$

$$0.02\left(C_3e^{-0.04t}+C_4e^{-0.12t}\right)-0.08\left(C_1e^{-0.04t}+C_2e^{-0.12t}\right)=\frac{d}{dt}\left(C_1e^{-0.04t}+C_2e^{-0.12t}\right) \quad --(A)$$

$$0.08\left(C_1e^{-0.04t}+C_2e^{-0.12t}\right)-0.08\left(C_3e^{-0.04t}+C_4e^{-0.12t}\right)=\frac{d}{dt}\left(C_3e^{-0.04t}+C_4e^{-0.12t}\right) \quad --(B)$$

Solve

$$\therefore \quad C_3=2C_1 \quad \& \quad C_4=-2C_2$$

$$\therefore \quad x=C_1e^{-0.04t}+C_2e^{-0.12t}$$

$$\therefore \quad y=2C_1e^{-0.04t}-2C_2e^{-0.12t}$$

$$B.C \quad t=0 \quad x=50, y=20$$

$$50=C_1+C_2$$

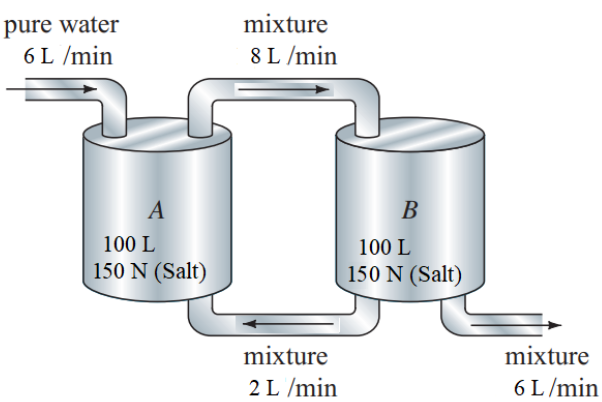
$$20=2C_1-2C_2$$

$$C_1=30 \qquad C_2=20$$

$$\therefore \quad x=30e^{-0.04t}+20e^{-0.12t}$$

$$\therefore \quad y=60e^{-0.04t}-40e^{-0.12t}$$

H.W.



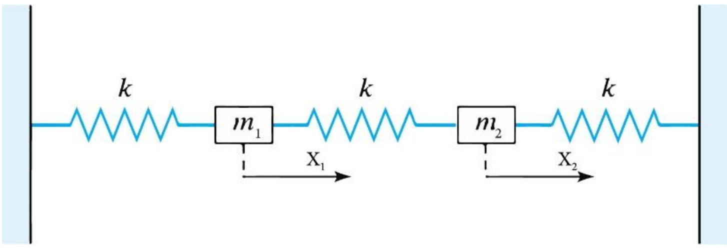
14.4- Mechanical Vibration

The system with two degree of freedom begins to max under the following conditions :

Initial displacement $X_1(0) = 1,\ X_2(0) = 1$.

Initial velocity $\overline{X_1}(0) = \sqrt{3k}\ ,\ \overline{X_2}(0) = \sqrt{3k}$.

Neglect the friction, derive the DE governing free vibration of system show below :



Mass*Acceleration = ∑F (in the direction of the motion)

$$K = \frac{EI}{L^3} \qquad (force/desp.)$$

$$F = KX \qquad , Mass \times Acceleration = \sum F$$

$Mass\,(1)$

$$1 \times \frac{d^2X_1}{dt^2} = K\,(X_2 - X_1) - KX_1$$

$$\frac{d^2X_1}{dt^2} + 2KX_1 - KX_2 = 0$$

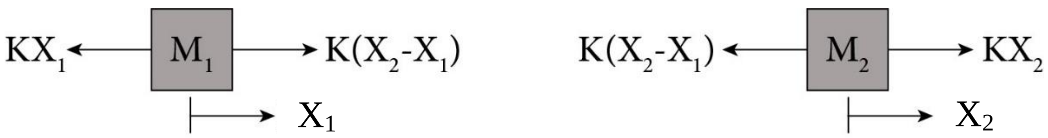
$$(D^2 + 2K)X_1 - KX_2 = 0 \qquad ---(1)$$

$Mass\,(2)$

$$1 \times \frac{d^2X_1}{dt^2} = -K\,(X_2 - X_1) - KX_1$$

$$\frac{d^2X_1}{dt^2} + 2KX_2 - KX_1 = 0$$

$$(D^2 + 2K)X_2 - KX_1 = 0 \qquad ---(2)$$



$$\begin{bmatrix} (D^2+2K) & -K \\ -K & (D^2+2K) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left\{(D^2+2K)^2 - K^2\right\}(X_1, X_2) = 0$$

$$(r^2+2K)^2 - K^2 = 0$$

$$(r^2+k)(r^2+3k) = 0$$

$$r_{1,2} = \mp\sqrt{k}\,i \qquad r_{3,4} = \mp\sqrt{3k}\,i$$

$$X_1 = C_1\cos\sqrt{k}\,t + C_2\sin\sqrt{k}\,t + C_3\cos\sqrt{3k}\,t + C_4\sin\sqrt{3k}\,t$$

$$X_2 = C_5\cos\sqrt{k}\,t + C_6\sin\sqrt{k}\,t + C_7\cos\sqrt{3k}\,t + C_8\sin\sqrt{3k}\,t$$

sub.into eq(1) & (2)

$$C_5 = C_1, C_6 = C_2, C_7 = -C_3 \;\&\; C_8 = -C_4$$

$$X_1 = C_1\cos\sqrt{k}\,t + C_2\sin\sqrt{k}\,t + C_3\cos\sqrt{3k}\,t + C_4\sin\sqrt{3k}\,t$$

$$X_2 = C_1\cos\sqrt{k}\,t + C_2\sin\sqrt{k}\,t - C_3\cos\sqrt{3k}\,t - C_4\sin\sqrt{3k}\,t$$

$$B.C \qquad t = 0 \qquad X_1 = X_2 = 1$$

$$1 = C_1 + C_3$$

$$1 = C_1 - C_3$$

$$C_1 = 1 \qquad C_3 = 0$$

$$\overline{X_1} = -C_1\sqrt{k}\sin\sqrt{k}\,t + C_2\sqrt{k}\cos\sqrt{k}\,t + C_4\sqrt{3k}\cos\sqrt{3k}\,t$$

$$\overline{X_2} = -C_1\sqrt{k}\sin\sqrt{k}\,t + C_2\sqrt{k}\cos\sqrt{k}\,t - C_4\sqrt{3k}\cos\sqrt{3k}\,t$$

$$B.C. \qquad \overline{X_1}(0) = \sqrt{3k},\; \overline{X_2}(0) = \sqrt{3k}$$

$$\sqrt{3k} = \sqrt{k}\,C_2 + \sqrt{3k}\,C_4$$

$$\sqrt{3k} = \sqrt{k}\,C_2 - \sqrt{3k}\,C_4$$

$$C_2 = \sqrt{3} \qquad C_4 = 0$$

$$X_1 = \cos\sqrt{k}\,t + \sqrt{3}\sin\sqrt{k}\,t$$

$$X_2 = \cos\sqrt{k}\,t + \sqrt{3}\sin\sqrt{k}\,t$$