Lectures of

Calculus II

Second Stage

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Chapter One

Matrices

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1- MATRICES

Matrix

Definition : A matrix is a rectangular array of numbers or functions (called elements), consisting of *m* rows and *n* columns:

$$A = [A]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

This is said to be a matrix of *order m* by *n*. For instance, here is a *2 by 3* matrix: $\begin{bmatrix} 4 & 6 & 2 \\ 7 & 5 & 9 \end{bmatrix}_{2\times 3}$

Column and Row Vectors

Definition : A m×1 matrix is called a column vector

$$A = \{A\}_{m\times 1} = \begin{Bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{Bmatrix}_{m\times 1} \quad , A = \begin{Bmatrix} 2 \\ 7 \\ 8 \\ 11 \\ -5 \end{Bmatrix} \quad , \mathrm{B} = \begin{Bmatrix} -3 \\ 4 \end{Bmatrix} , C = \begin{Bmatrix} 1 \\ 12 \\ 5 \end{Bmatrix}$$

A 1×n matrix is called a row vector

$$A = \{A\}_{1\times n} = \{a_{11} \quad a_{12} \quad a_{13} \quad \cdots \quad a_{1n}\}_{1\times n}$$

$$A = \{5 \quad 2 \quad 11 \quad -7 \quad 3\} \qquad B = \{-2 \quad 6\} \qquad C = \{3.5 \quad 8 \quad -4\}$$

Matrix Operations

1. **Matrix Addition** : When two matrices A and B are of the same size we can add them by adding their corresponding entries.
   If A and B are m×n matrices, then their sum is $A + B = (a_{ij} + b_{ij})_{mn}$

$$C_{ij} = A_{ij} + B_{ij}$$

$$A = [A]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \quad , \; B = [B]_{m\times n} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & b_{m3} & \cdots & b_{mn} \end{bmatrix}_{m\times n}$$

$$C = A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b & a_{23}+b_{23} & \cdots & a_{2n}+b_{2n} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} & \cdots & a_{3n}+b_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & a_{m3}+b_{m3} & \cdots & a_{mn}+b_{mn} \end{bmatrix}_{m\times n}$$

$$A + B = B + A$$

$$(A+B)+C = A+(B+C)$$

Example :

$$A = [A]_{3\times4} = \begin{bmatrix} 7 & -4 & 2 & 1 \\ 11 & 0 & 9 & -3 \\ 5 & 8 & -6 & 2 \end{bmatrix}_{3\times4} \quad , \quad B = [B]_{3\times4} = \begin{bmatrix} 2 & 8 & 1 & -5 \\ 4 & 5 & -4 & 7 \\ 3 & 1 & 9 & 13 \end{bmatrix}_{3\times4}$$

$$C = A + B = \begin{bmatrix} 7+2 & -4+8 & 2+1 & 1+(-5) \\ 11+4 & 0+5 & 9+(-4) & -3+7 \\ 5+3 & 8+1 & -6+9 & 2+13 \end{bmatrix}_{3\times4} = \begin{bmatrix} 9 & 4 & 3 & -4 \\ 15 & 5 & 5 & 4 \\ 8 & 9 & 3 & 15 \end{bmatrix}_{3\times4}$$

2. **Matrix Subtraction**: When two matrices A and B are of the same size we can add them by subtracting their corresponding entries.
   If A and B are m×n matrices, then their subtract is $A - B = (a_{ij} - b_{ij})_{mn}$

$$C_{ij} = A_{ij} - B_{ij}$$

$$A = [A]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \quad , \quad B = [B]_{m\times n} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & b_{m3} & \cdots & b_{mn} \end{bmatrix}_{m\times n}$$

$$C = A - B = \begin{bmatrix} a_{11}-b_{11} & a_{12}-b_{12} & a_{13}-b_{13} & \cdots & a_{1n}-b_{1n} \\ a_{21}-b_{21} & a_{22}-b & a_{23}-b_{23} & \cdots & a_{2n}-b_{2n} \\ a_{31}-b_{31} & a_{32}-b_{32} & a_{33}-b_{33} & \cdots & a_{3n}-b_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1}-b_{m1} & a_{m2}-b_{m2} & a_{m3}-b_{m3} & \cdots & a_{mn}-b_{mn} \end{bmatrix}_{m\times n}$$

$$A - B = -(B - A)$$

$$(A-B)-C = A-(B+C)$$

3. **Scalar Multiplication** : If k is a real number, then the scalar multiple of a matrix A is kA

$C_{ij} = kA_{ij}$

$$A = [A]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n} , \quad kA = k[A]_{m\times n} = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & ka_{23} & \cdots & ka_{2n} \\ ka_{31} & ka_{32} & ka_{33} & \cdots & ka_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ ka_{m1} & ka_{m2} & ka_{m3} & \cdots & ka_{mn} \end{bmatrix}_{m\times n}$$

If A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ Then 3A = $\begin{bmatrix} 3\times a_{11} & 3\times a_{12} \\ 3\times a_{21} & 3\times a_{22} \end{bmatrix}$

A = $\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ *then* 2A = $\begin{bmatrix} 8 & 6 \\ 4 & 2 \end{bmatrix}$ *and* 3A = $\begin{bmatrix} 12 & 9 \\ 6 & 3 \end{bmatrix}$

4. **Matrix Multiplication :** Let A be a matrix having m rows and p columns, and let B be a matrix having p rows and n columns.
The product AB is the m×n matrix

$$[A]_{r\times s} \times [B]_{s\times w} = [C]_{r\times w}$$

$$C_{ij} = \sum_{k=1}^{s} \left(a_{ik} \times b_{kj}\right)$$

Example :−

$$C_{3,2} = a_{3,1}\times b_{1,2} + a_{3,2}\times b_{2,2} + a_{3,3}\times b_{3,2} + a_{3,4}\times b_{4,2} + \dots\dots\dots + a_{3,s}\times b_{s,2}$$

$$A = [A]_{r\times s} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1s} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2s} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{r1} & a_{r2} & a_{r3} & \cdots & a_{rs} \end{bmatrix}_{r\times s} , \quad B = [B]_{s\times w} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1w} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2w} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3w} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{s1} & b_{s2} & b_{s3} & \cdots & b_{sw} \end{bmatrix}_{s\times w}$$

$$C_{ij} = \sum_{k=1}^{s} \left(a_{ik} \times b_{kj}\right)$$

$$C = A\times B = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1w} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2w} \\ c_{31} & c_{32} & c_{33} & \cdots & c_{3w} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{r1} & c_{r2} & c_{r3} & \cdots & c_{rw} \end{bmatrix}_{r\times w}$$

$$A=[A]_{4\times5}=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{bmatrix}_{4\times5} \quad , \quad B=[B]_{5\times2}=\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \\ b_{51} & b_{52} \end{bmatrix}_{5\times2}$$

$$C=A\times B=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{bmatrix}_{4\times\not5} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \\ b_{51} & b_{52} \end{bmatrix}_{\not5\times2}$$

$$[C]_{4\times2}=\begin{bmatrix} (a_{11}\times b_{11}+a_{12}\times b_{21}+a_{13}\times b_{31}+a_{14}\times b_{41}+a_{15}\times b_{51}) & (a_{11}\times b_{12}+a_{12}\times b_{22}+a_{13}\times b_{32}+a_{14}\times b_{42}+a_{15}\times b_{52}) \\ (a_{21}\times b_{11}+a_{22}\times b_{21}+a_{23}\times b_{31}+a_{24}\times b_{41}+a_{25}\times b_{51}) & (a_{21}\times b_{12}+a_{22}\times b_{22}+a_{23}\times b_{32}+a_{24}\times b_{42}+a_{25}\times b_{52}) \\ (a_{31}\times b_{11}+a_{32}\times b_{21}+a_{33}\times b_{31}+a_{34}\times b_{41}+a_{35}\times b_{51}) & (a_{31}\times b_{12}+a_{32}\times b_{22}+a_{33}\times b_{32}+a_{34}\times b_{42}+a_{35}\times b_{52}) \\ (a_{41}\times b_{11}+a_{42}\times b_{21}+a_{43}\times b_{31}+a_{44}\times b_{41}+a_{45}\times b_{51}) & (a_{41}\times b_{12}+a_{42}\times b_{22}+a_{43}\times b_{32}+a_{44}\times b_{42}+a_{45}\times b_{52}) \end{bmatrix}_{4\times2}$$

$$[C]_{4\times2}=\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \\ c_{41} & c_{42} \end{bmatrix}_{4\times2}$$

Note

$$A\times B\neq B\times A$$
$$(A\times B)\times C=A\times(B\times C)$$
$$(A+B)\times C=A\times C+B\times C$$

Example :-

$$A=[A]_{4\times5}=\begin{bmatrix} 3 & 5 & 1 & 2 & 7 \\ 4 & -1 & 3 & 6 & 5 \\ -2 & 9 & 6 & 3 & 4 \\ 5 & 4 & -3 & 0 & 2 \end{bmatrix}_{4\times5} \quad , \quad B=[B]_{5\times2}=\begin{bmatrix} 4 & 3 \\ 1 & -2 \\ 5 & 0 \\ 2 & 7 \\ 6 & -1 \end{bmatrix}_{5\times2}$$

$$C=A\times B$$

$$[C]_{4\times2}=\begin{bmatrix} (3\times4+5\times1+1\times5+2\times2+7\times6) & (3\times3+5\times(-2)+1\times0+2\times7+7\times(-1)) \\ (4\times4+(-1)\times1+3\times5+6\times2+5\times6) & (4\times3+(-1)\times(-2)+3\times0+6\times7+5\times(-1)) \\ (-2\times4+9\times1+6\times5+3\times2+4\times6) & (-2\times3+9\times(-2)+6\times0+3\times7+4\times(-1)) \\ (5\times4+4\times1+(-3)\times5+0\times2+2\times6) & (5\times3+4\times(-2)+(-3)\times0+0\times7+2\times(-1)) \end{bmatrix}_{4\times2}$$

$$[C]_{4\times2}=\begin{bmatrix} 68 & 6 \\ 72 & 51 \\ 61 & -7 \\ 21 & 5 \end{bmatrix}_{4\times2}$$

5. **Transposition of a Matrix :** Transposing a matrix means converting and m by n matrix into an n by m matrix, by “flipping” the rows and columns.

$A = [A]_{m\times n}$

$A^T = [A]_{n\times m}$

$a^T_{ij} = a_{ji}$

$(A+B)^T = A^T + B^T$

$$A = [A]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

$$A^T = [A^T]_{n\times m} = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{m2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{m3} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{bmatrix}_{n\times m}$$

Example :-

$$A = [A]_{6\times 4} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 \end{bmatrix}_{6\times 4} \quad , \quad A^T = [A^T]_{4\times 6} = \begin{bmatrix} 1 & 5 & 9 & 13 & 17 & 21 \\ 2 & 6 & 10 & 14 & 18 & 22 \\ 3 & 7 & 11 & 15 & 19 & 23 \\ 4 & 8 & 12 & 16 & 20 & 24 \end{bmatrix}_{4\times 6}$$

2- Special Types of Matrix

1. Zero (Null) Matrix

A zero, or null, matrix is one where every element is zero, e.g.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Square Matrix

A square matrix is one where the number of rows and columns are equal, e.g. a *2 by 2* matrix, a *3 by 3* matrix etc.

$$\begin{bmatrix} 5 & 3 \\ -2 & 1 \end{bmatrix} \quad \begin{bmatrix} 7 & 8 & 3 & 6 \\ 9 & 1 & 2 & 5 \\ 6 & 7 & 4 & 2 \\ 3 & 0 & 5 & 1 \end{bmatrix} \quad \begin{bmatrix} 4 & 6 & 2 \\ 7 & 5 & 9 \\ 3 & 8 & 1 \end{bmatrix}$$

3. Diagonal Matrix

A diagonal matrix is a square matrix in which all the elements are zero except for the elements on the leading diagonal, e.g.:

$$\begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \quad \begin{bmatrix} 8 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

4. Unit Matrix

A unit matrix is a square matrix in which all the elements on the leading diagonal are 1, and all the other elements are 0, e.g.:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A unit matrix is often denoted by $\boldsymbol{I}$ (identity matrix).

5. Null Matrix

A Null matrix is a square matrix in which all the elements all elements are 0, e.g.:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

6. Symmetric Matrix

A Symmetrix matrix is a square matrix where

$$a_{ij} = a_{ji}$$

for all elements. I.e., the matrix is symmetrical about the leading diagonal. For example

$$\begin{bmatrix} 3 & 5 \\ 5 & -2 \end{bmatrix} \quad \begin{bmatrix} 8 & 1 & 6 \\ 1 & 11 & -2 \\ 6 & -2 & 9 \end{bmatrix} \quad \begin{bmatrix} 7 & 8 & 4 & 1 \\ 8 & 5 & 3 & 0 \\ 4 & 3 & -4 & 9 \\ 1 & 0 & 9 & 3 \end{bmatrix}$$

7. Skew Symmetric Matrix

A skew symmetrix matrix is a square matrix where

$$a_{ij} = -a_{ji}$$

for all elements. Ie, the matrix is anti-symmetrical about the leading diagonal. This of course requires that elements along the diagonal must be zero. For example

$$\begin{bmatrix} 3 & 5 \\ -5 & -2 \end{bmatrix} \quad \begin{bmatrix} 8 & 1 & 6 \\ -1 & 11 & -2 \\ -6 & 2 & 9 \end{bmatrix} \quad \begin{bmatrix} 7 & 8 & 4 & 1 \\ -8 & 5 & -3 & 0 \\ -4 & 3 & -4 & 9 \\ -1 & 0 & -9 & 3 \end{bmatrix}$$

8. Orthogonal Matrix

An orthogonal matrix is a square matrix which product a unit matrix if it is multiplied by its own transpose. i.e.:

$$A \times A^T = I$$

➢ Exercise 1-1

H.W 1- : The matrices A to K are

$$A=\begin{bmatrix}3 & -1\\ -4 & 7\end{bmatrix}\qquad B=\begin{bmatrix}5 & 2\\ -1 & 6\end{bmatrix}\qquad D=\begin{bmatrix}-1.3 & 7.4\\ 2.5 & -3.9\end{bmatrix}$$

$$D=\begin{bmatrix}4 & -7 & 6\\ -2 & 4 & 0\\ 5 & 7 & -4\end{bmatrix}\qquad E=\begin{bmatrix}3 & 6 & 2\\ 5 & -3 & 7\\ -1 & 0 & 2\end{bmatrix}\qquad F=\begin{bmatrix}3.1 & 2.4 & 6.4\\ -1.6 & 3.8 & -1.9\\ 5.3 & 3.4 & -4.8\end{bmatrix}$$

$$G=\begin{bmatrix}6\\ -2\end{bmatrix}\qquad H=\begin{bmatrix}-2\\ 5\end{bmatrix}\qquad J=\begin{bmatrix}4\\ -11\\ 7\end{bmatrix}\qquad K=\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 0\end{bmatrix}$$

Evaluate these Matrices :

1- $A+B$
2- $D+E$
3- $A-B$
4- $A+B-C$
5- $5A+6B$
6- $2D+3E-4F$
7- $A\times H$
8- $A\times B$
9- $A\times C$
10- $D\times J$
11- $E\times K$
12- $D\times F$
13- Show that $A\times C\neq C\times A$

H.W 2- Determine the values of x and y for which the matrices are equal

1) $\begin{bmatrix}1 & x\\ y & -3\end{bmatrix}, \quad \begin{bmatrix}1 & y-2\\ 3x-2 & -3\end{bmatrix}$

2) $\begin{bmatrix}x^2 & 1\\ y & 5\end{bmatrix}, \quad \begin{bmatrix}9 & 1\\ 4x & 5\end{bmatrix}$

H.W 3- Find A^2 , B^2 and $A+B^T$

$$A=\begin{bmatrix}2 & -3\\ -5 & 4\end{bmatrix}\qquad B=\begin{bmatrix}-1 & 6\\ 3 & 2\end{bmatrix}$$

H.W 4- Verify that the quadratic form $ax^2+bxy+cy^2$ is the same as

$$\begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}a & \frac{b}{2}\\ \frac{b}{2} & c\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

H.W 5- Verify that $(AB)^T = B^T A^T$

$$A=\begin{bmatrix}3 & 2 & 4\\ 2 & 1 & 2\\ 4 & 2 & 3\end{bmatrix}\qquad B=\begin{bmatrix}2 & 1 & 3\\ 1 & 2 & 5\\ 0 & 2 & 1\end{bmatrix}$$

3- Determinants and Inverse Matrices

➢ Minor

A minor is the determinant of the square matrix formed by deleting one row and one column from some larger square matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \qquad m_{24} = \begin{bmatrix} 1 & 2 & 3 \\ 9 & 10 & 11 \\ 13 & 14 & 15 \end{bmatrix}$$

➢ Cofactors

The cofactor of an element is the minor multiplied by the appropriate sign

$c_{i1} = (-1)^{i+1} \times m_{i1}$ or more generally $\boxed{c_{ij} = (-1)^{i+j} \times m_{ij}}$

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

➢ Adjoint Matrices

Every square matrix has an adjoint matrix, found by taking the matrix of its cofactors, and transposing it.

$$A = [A]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

then the adjoint is

$$Adj\,(A) = \begin{bmatrix} c_{11} & c_{21} & c_{31} & \cdots & c_{n1} \\ c_{12} & c_{22} & c_{32} & \cdots & c_{n2} \\ c_{13} & c_{23} & c_{33} & \cdots & c_{n3} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{1m} & c_{2m} & c_{3m} & \cdots & c_{nm} \end{bmatrix}_{m\times n}$$

Determinants

A determinant is a function of a square matrix that reduces it to a single number.

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{vmatrix}$$

➢ Some Properties of Determinants

1. If A^T is the transpose of the n×n matrix A, then det A^T = det(A).
2. If any two rows (columns) of an n×n matrix A are the same, then det(A)=0.
3. If **B** is the matrix obtained by interchanging any two rows (columns) of an *n×n* matrix **A**, then det **B** =- det(A).
4. If B is the matrix obtained from an n×n matrix A by multiplying a row (column) by a nonzero real number k, then det B = k det (A).
5. If **A** and **B** are both *n×n* matrices, then det (**AB)=**det (**A)×**det (**B).**
6. Determinant Is Unchanged : Suppose B is the matrix obtained from an n×n matrix A by multiplying the entries in a row (column) by a nonzero real number k and adding the result to the corresponding entries in another row (column). Then det(B)= det(A).
7. Determinant of a Triangular Matrix Suppose A is an n×n triangular matrix (upper or lower). Then det(A)=$a_{11}a_{22} \ldots a_{nn}$, where $a_{11}, a_{22}, \ldots, a_{nn}$ are the entries on the main diagonal of A.

3.1.1 Determinant 2×2 matrix

$$[A]_{2\times2}=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad |A|=\begin{vmatrix} a & b \\ c & d \end{vmatrix}=ad-bc$$

$$[A]_{2\times2}=\begin{bmatrix} 3 & 5 \\ 7 & -2 \end{bmatrix}$$

$$|A|=\begin{vmatrix} 3 & 5 \\ 7 & -2 \end{vmatrix}=(3)(-2)-(5)(7)=-41$$

3.1.2 Determinant 3×3 matrix

$$[A]_{3\times3}=\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \qquad |A|=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\begin{matrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{matrix}$$

$$=(a_1b_2c_3+a_2b_3c_1+a_3b_1c_2)-(a_3b_2c_1+a_1b_3c_2+a_2b_1c_3)$$

$$[A]_{3\times3}=\begin{bmatrix} -1 & 4 & 3 \\ 2 & 6 & 4 \\ 3 & -2 & 8 \end{bmatrix} \qquad |A|=\begin{vmatrix} -1 & 4 & 3 \\ 2 & 6 & 4 \\ 3 & -2 & 8 \end{vmatrix}\begin{matrix} -1 & 4 \\ 2 & 6 \\ 3 & -2 \end{matrix}$$

$$=(-1\times6\times8+4\times4\times3+3\times2\times(-2))-(3\times6\times3+(-1)\times4\times(-2)+4\times2\times8)=-138$$

Example 2 (Adjoint Method):

$$A=[A]_{m\times n}=\begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & 2 \\ 2 & 1 & 5 \end{bmatrix}_{3\times3}$$

$$|A|=(+)\times2\times\begin{vmatrix} 2 & 3 & 5 \\ 1 & 4 & 2 \\ 2 & 1 & 5 \end{vmatrix}+(-)\times3\times\begin{vmatrix} 2 & 3 & 5 \\ 1 & 4 & 2 \\ 2 & 1 & 5 \end{vmatrix}+(+)\times5\times\begin{vmatrix} 2 & 3 & 5 \\ 1 & 4 & 2 \\ 2 & 1 & 5 \end{vmatrix}$$

$$|A|=(+)\times2\times\begin{vmatrix} 4 & 2 \\ 1 & 5 \end{vmatrix}+(-)\times3\times\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix}+(+)\times5\times\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix}$$

$$|A|=2\times(4\times5-2\times1)-3\times(1\times5-2\times2)+5\times(1\times1-4\times2)=-2$$

3.1.3 Determinant n×n matrix

1- (Adjoint Method)

$$A=[A]_{m\times n}=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

$$|A|=\sum_{i=1,\,j=any\,column}^{n}(-1)^{i+j}\times a_{ij}\times m_{ij} \qquad ex.\quad |A|=\sum_{i=1,\,j=1}^{n}(-1)^{i+1}\times a_{i1}\times m_{i1}$$

$$|A|=\sum_{j=1,i=any\,raw}^{m}(-1)^{i+j}\times a_{ij}\times m_{ij} \qquad ex.\quad |A|=\sum_{j=1,i=a1}^{m}(-1)^{1+j}\times a_{1j}\times m_{1j}$$

2- Reduction Method (Chio's condensation)

$$|A|_{m\times n}=\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}_{n\times n}=\frac{1}{(a_{11})^{n-2}}\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{31} & a_{3n} \end{vmatrix} \\ \vdots & \vdots & \vdots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{n1} & a_{n3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{vmatrix}_{n\times n}$$

3- Row Reduction Method (Gaussian Elimination)

$$|A|_{m\times n}=\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}_{n\times n} \Rightarrow |B|_{m\times n}=\begin{vmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ 0 & b_{22} & b_{23} & \cdots & b_{2n} \\ 0 & 0 & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & b_{nn} \end{vmatrix}_{n\times n}$$

$$|A|_{m\times n}=|B|_{m\times n}=diag(B)=b_{11}\times b_{22}\times b_{33}\ldots\ldots\times b_{nn}$$

Example :- *Find the determinant*

$$[A]=\begin{bmatrix}3&-2&2&1\\2&1&-2&3\\3&4&-8&1\\3&-11&12&2\end{bmatrix}_{4\times4}$$

Solution **:**

1- Adjoint Method

$$|A|=\begin{vmatrix}3&-2&2&1\\2&1&-2&3\\3&4&-8&1\\3&-11&12&2\end{vmatrix}$$

$$|A|=(+)\times(3)\times\begin{vmatrix}3&-2&2&1\\2&1&-2&3\\3&4&-8&1\\3&-11&12&2\end{vmatrix}+(-)\times(-2)\times\begin{vmatrix}3&-2&2&1\\2&1&-2&3\\3&4&-8&1\\3&-11&12&2\end{vmatrix}+(+)\times(2)\times\begin{vmatrix}3&-2&2&1\\2&1&-2&3\\3&4&-8&1\\3&-11&12&2\end{vmatrix}+(-)\times(1)\times\begin{vmatrix}3&-2&2&1\\2&1&-2&3\\3&4&-8&1\\3&-11&12&2\end{vmatrix}$$

$$|A|=3\times\begin{vmatrix}1&-2&3\\4&-8&1\\-11&12&2\end{vmatrix}+2\times\begin{vmatrix}2&-2&3\\3&-8&1\\3&12&2\end{vmatrix}+2\times\begin{vmatrix}2&1&3\\3&4&1\\3&-11&2\end{vmatrix}-\begin{vmatrix}2&1&-2\\3&4&-8\\3&-11&12\end{vmatrix}$$

$$M_1=\begin{vmatrix}1&-2&3\\4&-8&1\\-11&12&2\end{vmatrix}\begin{matrix}1&-2\\4&-8\\-11&12\end{matrix}=-110$$

$$M_2=\begin{vmatrix}2&-2&3\\3&-8&1\\3&12&2\end{vmatrix}\begin{matrix}2&-2\\3&-8\\3&12\end{matrix}=130$$

$$M_3=\begin{vmatrix}2&1&3\\3&4&1\\3&-11&2\end{vmatrix}\begin{matrix}2&1\\3&4\\3&-11\end{matrix}=-100$$

$$M_4=\begin{vmatrix}2&1&-2\\3&4&-8\\3&-11&12\end{vmatrix}\begin{matrix}2&1\\3&4\\3&-11\end{matrix}=-50$$

$$|A|=3\times(-110)+2\times(130)+2\times(-100)-(-50)=\boxed{-220}$$

2- Reduction Method

$$|A| = \frac{1}{(3)^{4-2}} \begin{vmatrix} \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} \\ \begin{vmatrix} 3 & -2 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 3 & -8 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 3 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & -2 \\ 3 & -11 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 3 & 12 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} \end{vmatrix} = \frac{1}{9} \begin{vmatrix} 7 & -10 & 7 \\ 18 & -30 & 0 \\ -27 & 30 & 3 \end{vmatrix}$$

$$|A| = \frac{1}{9} \times \frac{1}{(7)^{3-2}} \begin{vmatrix} \begin{vmatrix} 7 & -10 \\ 18 & -30 \end{vmatrix} & \begin{vmatrix} 7 & 7 \\ 18 & 0 \end{vmatrix} \\ \begin{vmatrix} 7 & -10 \\ -27 & 30 \end{vmatrix} & \begin{vmatrix} 7 & 7 \\ -27 & 3 \end{vmatrix} \end{vmatrix} = \frac{1}{9} \times \frac{1}{7} \times \begin{vmatrix} -30 & -126 \\ -60 & 210 \end{vmatrix}$$

$$|A| = \frac{1}{9} \times \frac{1}{7} \times (-13860) = \boxed{-220}$$

3- Row Reduction Method

$$\frac{-2}{3}R_1 + R_2 \Rightarrow \begin{vmatrix} 3 & -2 & 2 & 1 \\ 0 & \frac{7}{3} & \frac{-10}{3} & \frac{7}{3} \\ 3 & 4 & -8 & 1 \\ 3 & -11 & 12 & 2 \end{vmatrix} \qquad -R_1 + R_3 \Rightarrow \begin{vmatrix} 3 & -2 & 2 & 1 \\ 0 & \frac{7}{3} & \frac{-10}{3} & \frac{7}{3} \\ 0 & 6 & -10 & 0 \\ 3 & -11 & 12 & 2 \end{vmatrix}$$

$$-R_1 + R_4 \Rightarrow \begin{vmatrix} 3 & -2 & 2 & 1 \\ 0 & \frac{7}{3} & \frac{-10}{3} & \frac{7}{3} \\ 0 & 6 & -10 & 0 \\ 0 & -9 & 10 & 1 \end{vmatrix} \qquad -\frac{18}{7}R_2 + R_3 \Rightarrow \begin{vmatrix} 3 & -2 & 2 & 1 \\ 0 & \frac{7}{3} & \frac{-10}{3} & \frac{7}{3} \\ 0 & 0 & \frac{-10}{7} & -6 \\ 0 & -9 & 10 & 1 \end{vmatrix}$$

$$\frac{27}{7}R_2 + R_4 \Rightarrow \begin{vmatrix} 3 & -2 & 2 & 1 \\ 0 & \frac{7}{3} & \frac{-10}{3} & \frac{7}{3} \\ 0 & 0 & \frac{-10}{7} & -6 \\ 0 & 0 & \frac{-20}{7} & 10 \end{vmatrix} \qquad -2R_3 + R_4 \Rightarrow \begin{vmatrix} 3 & -2 & 2 & 1 \\ 0 & \frac{7}{3} & \frac{-10}{3} & \frac{7}{3} \\ 0 & 0 & \frac{-10}{7} & -6 \\ 0 & 0 & 0 & 22 \end{vmatrix}$$

$$|A| = (3) \times \left(\frac{7}{3}\right) \times \left(\frac{-10}{7}\right) \times (22) = \boxed{-220}$$

4- The Inverse of a Matrix

The inverse (or reciprocal) of a square matrix is denoted by the A^{-1}

4.1.1 Adjoint Method

$$A^{-1}=\frac{1}{|A|}adj(A)$$

$$Adj(A)=\begin{bmatrix} c_{11} & c_{21} & c_{31} & \cdots & c_{n1} \\ c_{12} & c_{22} & c_{32} & \cdots & c_{n2} \\ c_{13} & c_{23} & c_{33} & \cdots & c_{n3} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{1m} & c_{2m} & c_{3m} & \cdots & c_{nm} \end{bmatrix}_{m\times n}$$

$$c_{ij}=(-1)^{i+j}\times m_{ij}$$

$$m_{ij}=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}_{(m-1)\times(n-1)}$$

4.1.2 Row Reduction Method (Gaussian Elimination)

$$\left(A|I\right)\Rightarrow\left(I|A^{-1}\right)$$

$$\left(\begin{array}{ccccc|ccccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & 1 & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & 0 & 1 & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & 0 & 0 & 0 & \cdots & 1 \end{array}\right)\Rightarrow\left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & \cdots & 0 & b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ 0 & 1 & 0 & \cdots & 0 & b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ 0 & 0 & 1 & \cdots & 0 & b_{31} & b_{32} & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \end{array}\right)$$

$$\therefore [A]^{-1}=\begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \end{bmatrix}$$

Example :- Find the inverse matrix

$$[A]=\begin{bmatrix}2&0&1\\-2&3&4\\-5&5&6\end{bmatrix}_{3\times3}$$

Solution :

1- Adjoint Method

$$[A]=\begin{bmatrix}2&0&1\\-2&3&4\\-5&5&6\end{bmatrix}_{3\times3}$$

$$c_{11}=+\begin{vmatrix}3&4\\5&6\end{vmatrix}=-2 \qquad c_{12}=-\begin{vmatrix}-2&4\\-5&6\end{vmatrix}=-8 \qquad c_{13}=+\begin{vmatrix}-2&3\\-5&5\end{vmatrix}=5$$

$$c_{21}=-\begin{vmatrix}0&1\\5&6\end{vmatrix}=5 \qquad c_{22}=+\begin{vmatrix}2&1\\-5&6\end{vmatrix}=17 \qquad c_{23}=-\begin{vmatrix}2&0\\-5&5\end{vmatrix}=-10$$

$$c_{31}=+\begin{vmatrix}0&1\\3&4\end{vmatrix}=-3 \qquad c_{32}=-\begin{vmatrix}2&1\\-2&4\end{vmatrix}=-10 \qquad c_{33}=+\begin{vmatrix}2&0\\-2&3\end{vmatrix}=6$$

$$[C]=\begin{bmatrix}-2&-8&5\\5&17&-10\\-3&-10&6\end{bmatrix}$$

$$adj(A)=[C]^T=\begin{bmatrix}-2&5&-3\\-8&17&-10\\5&-10&6\end{bmatrix}$$

$$|A|=2\times(-2)-(0)\times(-8)+(1)\times(5)=1$$

$$[A]^{-1}=\frac{1}{|A|}adj(A)=\frac{1}{1}\begin{bmatrix}-2&5&-3\\-8&17&-10\\5&-10&6\end{bmatrix}$$

$$[A]^{-1}=\begin{bmatrix}-2&5&-3\\-8&17&-10\\5&-10&6\end{bmatrix}$$

2- Row Reduction Method

$$[A]=\begin{bmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{bmatrix}_{3\times 3}$$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ -2 & 3 & 4 & 0 & 1 & 0 \\ -5 & 5 & 6 & 0 & 0 & 1 \end{array}\right) \Rightarrow \xrightarrow{R_1+R_2} \left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ -5 & 5 & 6 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\frac{5}{2}R_1+R_3} \left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 5 & \frac{17}{2} & \frac{5}{2} & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-\frac{5}{3}R_2+R_3} \left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 0 & \frac{1}{6} & \frac{5}{6} & \frac{-5}{3} & 1 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_1,\ \frac{1}{3}R_2,\ 6R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{5}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 5 & -10 & 6 \end{array}\right)$$

$$\xrightarrow{-\frac{5}{3}R_3+R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -8 & 17 & -10 \\ 0 & 0 & 1 & 5 & -10 & 6 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{2}R_3+R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 5 & -3 \\ 0 & 1 & 0 & -8 & 17 & -10 \\ 0 & 0 & 1 & 5 & -10 & 6 \end{array}\right)$$

$$\therefore [A]^{-1}=\begin{bmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{bmatrix}$$

➢ *Exercise 1-2*

H.W 1- Evaluate the determinant of the given matrix

$\begin{bmatrix} -2 & 5 \\ 3 & -6 \end{bmatrix}$ $\begin{bmatrix} 2j & -3j \\ 1+j & j \end{bmatrix}$

$\begin{bmatrix} 3 & 0 & 2 \\ 2 & 7 & 1 \\ 2 & 6 & 4 \end{bmatrix}$ $\begin{bmatrix} 1 & -1 & -1 \\ 2 & 2 & -2 \\ 1 & 1 & 9 \end{bmatrix}$ $\begin{bmatrix} 3 & 5 & 1 \\ -1 & 2 & 5 \\ 7 & -4 & 10 \end{bmatrix}$ $\begin{bmatrix} a & a+1 & a+2 \\ b & b+1 & b+2 \\ c & c+1 & c+2 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & -3 & 0 \\ 1 & 5 & 3 & 2 \\ 1 & -2 & 1 & 0 \\ 4 & 8 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 2 & 1 & -2 & 1 \\ 0 & 5 & 0 & 4 \\ 1 & 6 & 1 & 0 \\ 5 & -1 & 1 & 1 \end{bmatrix}$

$\begin{bmatrix} 2 & 2 & 0 & 0 & -2 \\ 1 & 1 & 6 & 0 & 5 \\ 1 & 0 & 2 & -1 & -1 \\ 2 & 0 & 1 & -2 & 3 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$

H.W 2- Show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$

H.W 3- verify that the matrix **B** is the inverse of the matrix **A**.

$$A = \begin{bmatrix} 1 & \frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 & 2 \\ 1 & -1 & 2 \\ -3 & 2 & -3 \end{bmatrix}$$

H.W 4- Evaluate the inverse matrix

$\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}$ $\begin{bmatrix} 6 & 0 \\ -3 & 2 \end{bmatrix}$ $\begin{bmatrix} -2\pi & -\pi \\ -\pi & \pi \end{bmatrix}$ $\begin{bmatrix} \sin x & \cos x \\ -\cos x & \sin x \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 4 \\ 1 & -1 & 1 \end{bmatrix}$ $\begin{bmatrix} 2 & 3 & 0 \\ 0 & 11 & 14 \\ -1 & 4 & 7 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ -1 & 5 & 1 \end{bmatrix}$

H.W 5- Find the value of constant ***a***

$$\begin{vmatrix} 8 & -2 & -10 \\ 2 & a & -2 \\ 6 & 3 & 8 \end{vmatrix} = -328 \qquad \begin{vmatrix} 4 & a & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{vmatrix} = -212$$

5- Solving Simultaneous Algebraic Equations

There are various methods to solve a set of algebraic equations, such as :

1. Cramer's rule.
2. Gauss elimination.
3. Gauss-Jordan elimination.
4. and matrix inverse.

Cramer's rule

$$A\times X=B \qquad \Rightarrow \qquad X_i=\frac{|A_i|}{|A|}.$$

Example: Solve the following set of algebraic equations

$$2x+4y-2z-2=0$$
$$x+z-3=0$$
$$2x+y-z-1=0$$

Solution :

$$\begin{array}{l}2x+4y-2z-2=0\\ x+z-3=0\\ 2x+y-z-1=0\end{array} \quad \Rightarrow \begin{bmatrix}2 & 4 & -2\\ 1 & 0 & 1\\ 2 & 1 & -1\end{bmatrix}\begin{Bmatrix}x\\ y\\ z\end{Bmatrix}=\begin{bmatrix}2\\ 3\\ 1\end{bmatrix}$$

$$|A|=\begin{vmatrix}2 & 4 & -2\\ 1 & 0 & 1\\ 2 & 1 & -1\end{vmatrix}=2\begin{vmatrix}0 & 1\\ 1 & -1\end{vmatrix}+(-1)(4)\begin{vmatrix}1 & 1\\ 2 & -1\end{vmatrix}+(-2)\begin{vmatrix}1 & 0\\ 2 & 1\end{vmatrix}=8$$

$$|A_1|=\begin{vmatrix}2 & 4 & -2\\ 3 & 0 & 1\\ 1 & 1 & -1\end{vmatrix}=2\begin{vmatrix}0 & 1\\ 1 & -1\end{vmatrix}+(-1)(4)\begin{vmatrix}3 & 1\\ 1 & -1\end{vmatrix}+(-2)\begin{vmatrix}3 & 0\\ 1 & 1\end{vmatrix}=8$$

$$|A_2|=\begin{vmatrix}2 & 2 & -2\\ 1 & 3 & 1\\ 2 & 1 & -1\end{vmatrix}=2\begin{vmatrix}3 & 1\\ 1 & -1\end{vmatrix}+(-1)(2)\begin{vmatrix}1 & 1\\ 2 & -1\end{vmatrix}+(-2)\begin{vmatrix}1 & 3\\ 2 & 1\end{vmatrix}=8$$

$$|A_3|=\begin{vmatrix}2 & 4 & 2\\ 1 & 0 & 3\\ 2 & 1 & 1\end{vmatrix}=2\begin{vmatrix}0 & 3\\ 1 & 1\end{vmatrix}+(-1)(4)\begin{vmatrix}1 & 3\\ 2 & 1\end{vmatrix}+2\begin{vmatrix}1 & 0\\ 2 & 1\end{vmatrix}=16$$

$$x=\frac{|A_1|}{|A|}=\frac{8}{8}=1 \qquad ,y=\frac{|A_2|}{|A|}=\frac{8}{8}=1 \qquad ,z=\frac{|A_3|}{|A|}=\frac{16}{8}=2$$

Gauss elimination

Example: Solve the following set of algebraic equations

$$2x+4y-2z-2=0$$
$$x+z-3=0$$
$$2x+y-z-1=0$$

Solution :

$$\begin{aligned}2x+4y-2z-2&=0\\x+z-3&=0\\2x+y-z-1&=0\end{aligned} \Rightarrow \begin{bmatrix}2&4&-2\\1&0&1\\2&1&-1\end{bmatrix}\begin{Bmatrix}x\\y\\z\end{Bmatrix}=\begin{bmatrix}2\\3\\1\end{bmatrix}$$

$$\begin{bmatrix}2&4&-2\\1&0&1\\2&1&-1\end{bmatrix}\begin{Bmatrix}x\\y\\z\end{Bmatrix}=\begin{bmatrix}2\\3\\1\end{bmatrix}\xrightarrow{R_1/2}\begin{bmatrix}1&2&-1\\1&0&1\\2&1&-1\end{bmatrix}\begin{Bmatrix}x\\y\\z\end{Bmatrix}=\begin{bmatrix}1\\3\\1\end{bmatrix}$$

$$\xrightarrow[-2R_1+R_3]{-R_1+R_2}\begin{bmatrix}1&2&-1\\0&-2&2\\0&-3&1\end{bmatrix}\begin{Bmatrix}x\\y\\z\end{Bmatrix}=\begin{bmatrix}1\\2\\-1\end{bmatrix}$$

$$\xrightarrow{R_2/(-2)}\begin{bmatrix}1&2&-1\\0&1&-1\\0&-3&1\end{bmatrix}\begin{Bmatrix}x\\y\\z\end{Bmatrix}=\begin{bmatrix}1\\-1\\-1\end{bmatrix}$$

$$\xrightarrow{3R_2+R_3}\begin{bmatrix}1&2&-1\\0&1&-1\\0&0&-2\end{bmatrix}\begin{Bmatrix}x\\y\\z\end{Bmatrix}=\begin{bmatrix}1\\-1\\-4\end{bmatrix}$$

$$\xrightarrow{R_3/(-2)}\begin{bmatrix}1&2&-1\\0&1&-1\\0&0&1\end{bmatrix}\begin{Bmatrix}x\\y\\z\end{Bmatrix}=\begin{bmatrix}1\\-1\\2\end{bmatrix}$$

$$\begin{bmatrix}1&2&-1\\0&1&-1\times 2\\0&0&1\end{bmatrix}\begin{Bmatrix}x\\y\\2\end{Bmatrix}=\begin{bmatrix}1\\-1\\2\end{bmatrix}\Rightarrow\begin{bmatrix}1&2\times 1&-1\times 2\\0&1&-1\\0&0&1\end{bmatrix}\begin{Bmatrix}x\\1\\2\end{Bmatrix}=\begin{bmatrix}1\\-1\\2\end{bmatrix}\Rightarrow\begin{bmatrix}1&2&-1\\0&1&-1\\0&0&1\end{bmatrix}\begin{Bmatrix}1\\1\\2\end{Bmatrix}=\begin{bmatrix}1\\-1\\2\end{bmatrix}$$

$$\therefore\begin{Bmatrix}x\\y\\z\end{Bmatrix}=\begin{bmatrix}1\\1\\2\end{bmatrix}$$

Gauss-Jordan elimination

Example: Solve the following set of algebraic equations

$$2x+4y-2z-2=0$$
$$x+z-3=0$$
$$2x+y-z-1=0$$

Solution :

$$\begin{aligned}2x+4y-2z-2&=0\\ x+z-3&=0\\ 2x+y-z-1&=0\end{aligned} \Rightarrow \begin{bmatrix}2&4&-2\\1&0&1\\2&1&-1\end{bmatrix}\begin{Bmatrix}x\\y\\z\end{Bmatrix}=\begin{bmatrix}2\\3\\1\end{bmatrix} \xrightarrow{R_1/2} \begin{bmatrix}1&2&-1\\1&0&1\\2&1&-1\end{bmatrix}\begin{Bmatrix}x\\y\\z\end{Bmatrix}=\begin{bmatrix}1\\3\\1\end{bmatrix}$$

$$\begin{matrix}\xrightarrow{-R_1+R_2}\\ \xrightarrow{-2R_1+R_3}\end{matrix}\begin{bmatrix}1&2&-1\\0&-2&2\\0&-3&1\end{bmatrix}\begin{Bmatrix}x\\y\\z\end{Bmatrix}=\begin{bmatrix}1\\2\\-1\end{bmatrix}$$

$$\xrightarrow{R_2/(-2)}\begin{bmatrix}1&2&-1\\0&1&-1\\0&-3&1\end{bmatrix}\begin{Bmatrix}x\\y\\z\end{Bmatrix}=\begin{bmatrix}1\\-1\\-1\end{bmatrix}$$

$$\xrightarrow{3R_2+R_3}\begin{bmatrix}1&2&-1\\0&1&-1\\0&0&-2\end{bmatrix}\begin{Bmatrix}x\\y\\z\end{Bmatrix}=\begin{bmatrix}1\\-1\\-4\end{bmatrix}$$

$$\xrightarrow{R_3/(-2)}\begin{bmatrix}1&2&-1\\0&1&-1\\0&0&1\end{bmatrix}\begin{Bmatrix}x\\y\\z\end{Bmatrix}=\begin{bmatrix}1\\-1\\2\end{bmatrix}$$

$$\xrightarrow{R_3+R_2}\begin{bmatrix}1&2&-1\\0&1&0\\0&0&1\end{bmatrix}\begin{Bmatrix}x\\y\\z\end{Bmatrix}=\begin{bmatrix}1\\1\\2\end{bmatrix}$$

$$\xrightarrow{R_3+R_1}\begin{bmatrix}1&2&0\\0&1&0\\0&0&1\end{bmatrix}\begin{Bmatrix}x\\y\\z\end{Bmatrix}=\begin{bmatrix}3\\1\\2\end{bmatrix}$$

$$\xrightarrow{-2R_2+R_1}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{Bmatrix}x\\y\\z\end{Bmatrix}=\begin{bmatrix}1\\1\\2\end{bmatrix} \Rightarrow \therefore\begin{Bmatrix}x\\y\\z\end{Bmatrix}=\begin{bmatrix}1\\1\\2\end{bmatrix}$$

Inverse of a matrix

$$A\times X=B \qquad \Rightarrow \qquad X=A^{-1}\times B$$

Example: Solve the following set of algebraic equations

$$2x+4y-2z-2=0$$
$$x+z-3=0$$
$$2x+y-z-1=0$$

Solution :

$$\begin{aligned}2x+4y-2z-2&=0\\x+z-3&=0\\2x+y-z-1&=0\end{aligned} \Rightarrow \begin{bmatrix}2&4&-2\\1&0&1\\2&1&-1\end{bmatrix}\begin{Bmatrix}x\\y\\z\end{Bmatrix}=\begin{bmatrix}2\\3\\1\end{bmatrix} \Rightarrow [A]=\begin{bmatrix}2&4&-2\\1&0&1\\2&1&-1\end{bmatrix}_{3\times3}$$

$$c_{11}=+\begin{vmatrix}0&1\\1&-1\end{vmatrix}=-1 \qquad c_{12}=-\begin{vmatrix}1&1\\2&-1\end{vmatrix}=3 \qquad c_{13}=+\begin{vmatrix}1&0\\2&1\end{vmatrix}=1$$

$$c_{21}=-\begin{vmatrix}4&-2\\1&-1\end{vmatrix}=2 \qquad c_{22}=+\begin{vmatrix}2&-2\\2&-1\end{vmatrix}=2 \qquad c_{23}=-\begin{vmatrix}2&4\\2&1\end{vmatrix}=6$$

$$c_{31}=+\begin{vmatrix}4&-2\\0&1\end{vmatrix}=4 \qquad c_{32}=-\begin{vmatrix}2&-2\\1&1\end{vmatrix}=-4 \qquad c_{33}=+\begin{vmatrix}2&4\\1&0\end{vmatrix}=-4$$

$$[C]=\begin{bmatrix}-1&3&1\\2&2&6\\4&-4&-4\end{bmatrix}$$

$$|A|=2\times(-1)+4\times3+(-2)\times1=8$$

$$adj(A)=[C]^T=\begin{bmatrix}-1&2&4\\3&2&-4\\1&6&-4\end{bmatrix}$$

$$[A]^{-1}=\frac{1}{|A|}adj(A)=\frac{1}{8}\begin{bmatrix}-1&2&4\\3&2&-4\\1&6&-4\end{bmatrix}$$

$$X=A^{-1}\times B=\frac{1}{8}\begin{bmatrix}-1&2&4\\3&2&-4\\1&6&-4\end{bmatrix}\times\begin{bmatrix}2\\3\\1\end{bmatrix}$$

$$\begin{Bmatrix}x\\y\\z\end{Bmatrix}=\frac{1}{8}\begin{Bmatrix}-1\times2+2\times3+4\times1=8\\3\times2+2\times3-4\times1=8\\1\times2+6\times3-4\times1=16\end{Bmatrix}$$

$$\begin{Bmatrix}x\\y\\z\end{Bmatrix}=\begin{Bmatrix}8/8\\8/8\\16/8\end{Bmatrix} \Rightarrow \therefore\begin{Bmatrix}x\\y\\z\end{Bmatrix}=\begin{Bmatrix}1\\1\\2\end{Bmatrix}$$

Exercise 1-3

H.W 1- use an inverse matrix to solve the given system of equations.

1) $x_1+x_2=4$, $2x_1-x_2=14$

2) $x_1+2x_2=4$, $3x_1+4x_2=-3$

3) $x_1+x_3=-4$, $x_1+x_2+x_3=0$, $5x_1-x_2=6$

4) $x_1-x_2+x_3=1$, $2x_1+x_2+2x_3=2$, $3x_1+2x_2-x_3=-3$

5) $x_1-x_3=2$, $x_2+x_3=1$, $-x_1+x_2+2x_3+x_4=-5$, $x_3-x_4=3$

H.W 2- Solve the given system of equations by Cramer's rule.

1) $x_1+x_2=4$, $2x_1-x_2=2$

2) $0.21x_1+0.57x_2=0.369$, $0.1x_1+0.2x_2=0.135$

3) $x_1-x_2-3x_3=3$, $x_1+x_2-x_3=5$, $3x_1+2x_2=-4$

4) $u+2v+w=8$, $2u-2v+2w=7$, $u-4v+3w=1$

5) $x_1-x_3=2$, $x_2+x_3=1$, $-x_1+x_2+2x_3+x_4=-5$, $x_3-x_4=3$

H.W 3- Solve the given system of equations by Gauss-Jordan elimination.

$$\begin{aligned}2x+4y-2z-2&=0\\ x+z-3&=0\\ 2x+y-z-1&=0\end{aligned}$$

$$\begin{aligned}x+y+z&=6\\ 2x+3y+4z&=20\\ 4x+2y+3z&=12\end{aligned}$$

$$\begin{aligned}5x+2y+z&=8\\ 3x+2y&=5\\ x+2z&=3\end{aligned}$$

H.W 4- Solve the given system of equations by Gauss elimination.

$$\begin{bmatrix}2&3&0&5\\1&4&0&2\\5&4&8&5\\2&1&0&5\end{bmatrix}\begin{Bmatrix}x_1\\x_2\\x_3\\x_4\end{Bmatrix}=\begin{Bmatrix}4\\12\\5\\7\end{Bmatrix}$$

$$\begin{bmatrix}0&3&0&-2\\1&4&0&2\\3&2&6&1\\5&0&3&4\end{bmatrix}\begin{Bmatrix}x_1\\x_2\\x_3\\x_4\end{Bmatrix}=\begin{Bmatrix}8\\0\\-3\\9\end{Bmatrix}$$

H.W 5- Using matrices, solve the equations for I_1, I_2 and I_3.

$$\begin{aligned}2.4I_1+3.6I_2+4.8I_3&=1.2\\ -3.9I_1+1.3I_2-6.5I_3&=2.6\\ 1.7I_1+11.9I_2+8.5I_3&=0\end{aligned}$$

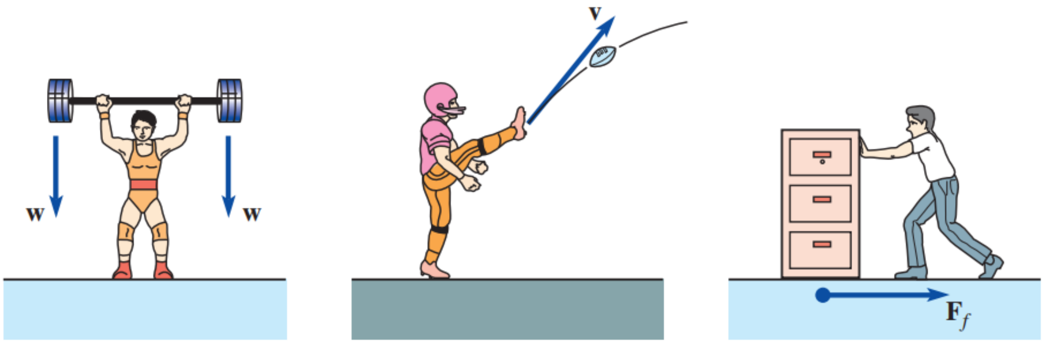
Chapter Two

Vectors

Mr. Munther 2020-2021
Civil Engineering Department

1- Vectors in 2-Space

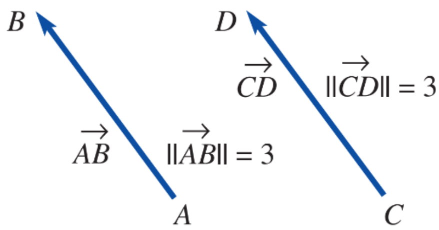
In science, mathematics, and engineering, we distinguish two important quantities: ***scalars*** and ***vectors***. A scalar is simply a real number or a quantity that has **magnitude**. For example, length, temperature, and concrete pressure are represented by numbers such as 80 m, 20C, and 27 N/mm^2. A vector, on the other hand, is usually described as a quantity that has both **magnitude** and **direction**.



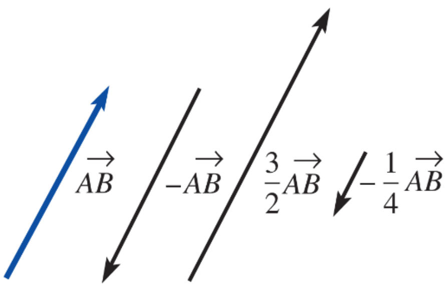
Geometric Vectors

Geometrically, a vector can be represented by a directed line segment—that is, by an arrow—and is denoted either by a boldface symbol or by a symbol with an arrow over it; for example, $\vec{v}, \vec{u}$ or $\overrightarrow{AB}$

A vector whose initial point (or end) is *A* and whose terminal point (or tip) is *B* is written $\overrightarrow{AB}$. The magnitude of a vector is written $\|AB\|$. Two vectors that have the same magnitude and same direction are said to be equal. Thus, we have $\overrightarrow{AB} = \overrightarrow{CD}$. Vectors are said to be **free**, which means that a vector can be moved from one position to another provided its magnitude and direction are not changed.



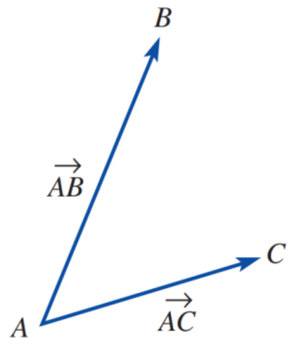
The negative of a vector $\overrightarrow{AB}$, written $-\overrightarrow{AB}$, is a vector that has the same magnitude as $\overrightarrow{AB}$ but is opposite in direction.

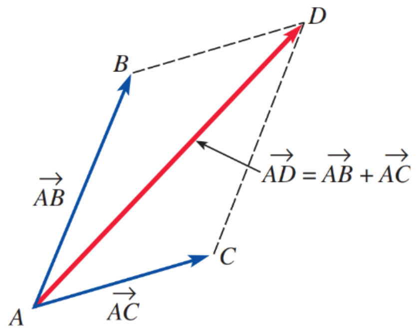


➢ Addition and Subtraction

Two vectors can be considered as having a common initial point, such as A in figure. Thus, if nonparallel vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are the sides of a parallelogram as in figure below, we say the vector that is the main diagonal, or $\overrightarrow{AD}$, is the sum of $\overrightarrow{AB}$ and $\overrightarrow{AC}$. We write

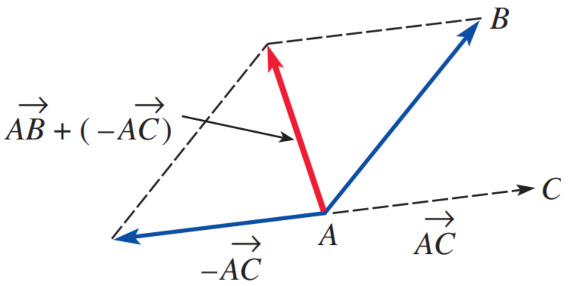
$$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{AC}$$

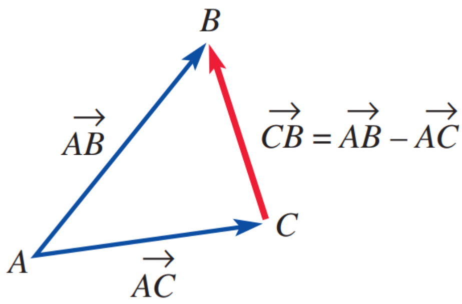




The difference of two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is defined by

$$\overrightarrow{CB} = \overrightarrow{AB} - \overrightarrow{AC} = \overrightarrow{AB} + \left(-\overrightarrow{AC}\right)$$

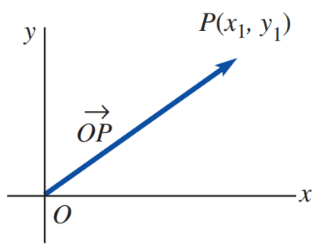




If $\overrightarrow{AB} = \overrightarrow{AC}$, then $\overrightarrow{AB} - \overrightarrow{AC} = 0$

2- Vectors in a Coordinate Plane

The vector shown in figure, with initial point the origin O and terminal point P(x_1, y_1), is called the *position vector* of the point P and is written

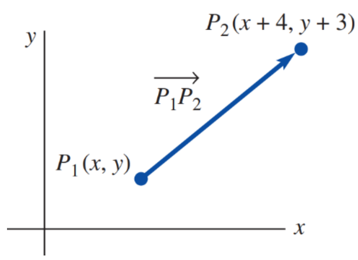


$$\overrightarrow{OP} = \langle x_1, y_1 \rangle$$

In general, a vector $\vec{A}$ is any ordered pair of real numbers

$$\vec{A} = \langle a_1, a_2 \rangle$$

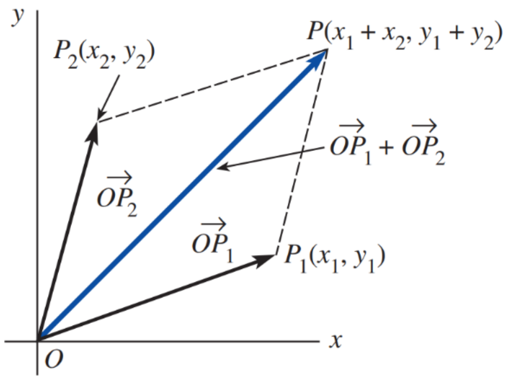
The numbers a_1 and a_2 are said to be the *components* of the vector $\vec{A}$.

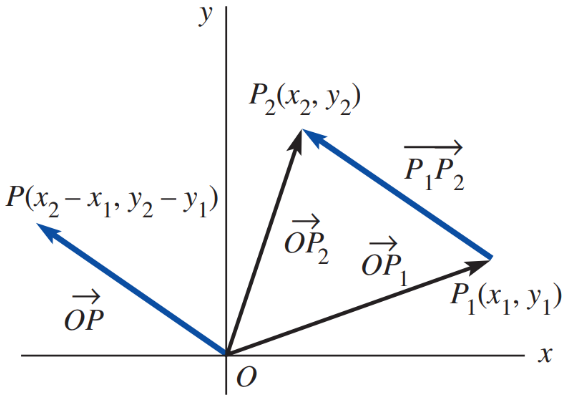


$$\overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1 \rangle$$

Definition

Addition	$a+b=\langle a_1+b_1,a_2+b_2\rangle$	$a-b=\langle a_1-b_1,a_2-b_2\rangle$
Scalar Multiplication	$ka=\langle ka_1,ka_2\rangle$	
Equality	$a=b$ if and only if $a_1=b_1,\ a_2=b_2$	





Example [1] If $a\langle 1,4\rangle$ and $b\langle -6,3\rangle$, find $a+b, a-b$, and $2a+3b$.

Solution

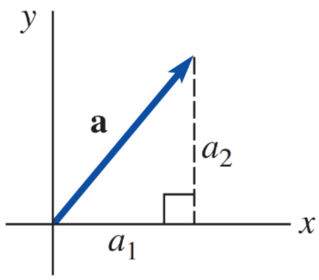
$$a+b=\langle a_1+b_1,a_2+b_2\rangle=\langle 1+(-6),4+3\rangle=\langle -5,7\rangle$$
$$a-b=\langle a_1-b_1,a_2-b_2\rangle=\langle 1-(-6),4-3\rangle=\langle 7,1\rangle$$
$$2a+3b=\langle 2a_1+3b_1,2a_2+3b_2\rangle=\langle 2\times 1+3\times(-6),2\times 4+3\times 3\rangle=\langle -16,17\rangle$$

The zero vector 0 is defined as $0=\langle 0,0\rangle$.

Magnitude

The magnitude, length, or norm of a vector a is denoted by $\|a\|$. Motivated by the Pythagorean theorem and figure, we define the magnitude of a vector :

$$\|a\|=\sqrt{a_1^2+a_2^2}$$



Unit Vectors

A vector that has magnitude 1 is called a unit vector. The *normalization* of the vector a is a unit vector

$$u=\frac{1}{\|a\|}a \qquad \text{where } \|u\|=1$$

Example [2] Given $a\langle 2,-1\rangle$, form a unit vector in the same direction as **a**. In the opposite direction of **a**.

Solution

$$\|a\| = \sqrt{a_1^2 + a_2^2} = \sqrt{2^2 + (-1)^2} = \sqrt{5}$$

$$u = \frac{1}{\|a\|}\, a = \frac{1}{\sqrt{5}}\langle 2,-1\rangle = \left\langle \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \right\rangle$$

$$-u = -1\left\langle \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \right\rangle = \left\langle \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

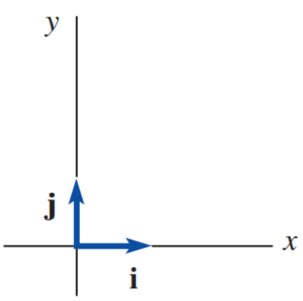
➢ The i, j Vectors

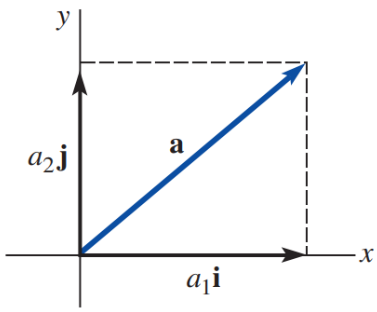
Any vector $a = \langle a_1, a_2\rangle$ can be written as a sum :

$$\langle a_1, a_2\rangle = \langle a_1, 0\rangle + \langle 0, a_2\rangle = a_1\langle 1,0\rangle + a_2\langle 0,1\rangle$$

The unit vectors $\langle 1,0\rangle$ and $\langle 0,1\rangle$ are usually given the special symbols **i** and **j** . Thus if :

$i\ \langle 1,0\rangle$ $\quad$ $j = \langle 0,1\rangle$ $\quad$ become $\quad$ $\vec{A} = a_1 i + a_2 j$



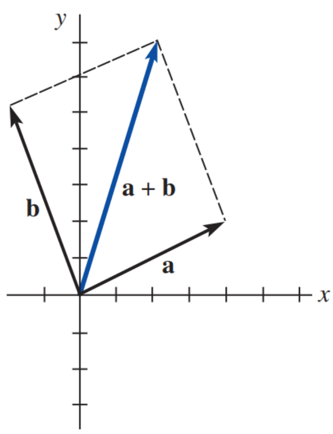


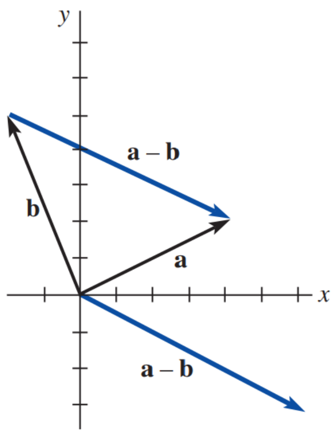
Example [3] Let $\vec{a} = 4i + 2j$ and $\vec{b} = -2i + 5j$. Graph $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$.

Solution

$$\vec{a} + \vec{b} = (4 + (-2))i + (2+5)j = 2i + 7j$$

$$\vec{a} - \vec{b} = (4 - (-2))i + (2-5)j = 6i - 3j$$





➢ Exercises 2-1

H.W 1- Find (1) $3\vec{a}$,(2) $\vec{a}+\vec{b}$,(3) $\vec{a}-\vec{b}$,(4) $\|\vec{a}+\vec{b}\|$ and (5) $\|\vec{a}-\vec{b}\|$ if :

1. $\vec{a}=2i+4j$, $\vec{b}=-i+4j$
2. $\vec{a}=-3i+2j$, $\vec{b}=7j$
3. $\vec{a}=-\vec{b}$, $\vec{b}=2i-9j$
4. $\vec{a}=\langle 1,3\rangle$, $\vec{b}=-5\vec{a}$
5. $\vec{a}=\langle 4,10\rangle$, $\vec{b}=-2\langle 1,3\rangle$

H.W 2- Find the vector $\overrightarrow{P_1P_2}$

1. $P_1(3,2)$, $P_2(5,7)$
2. $P_1(-2,-1)$, $P_2(4,-5)$
3. $P_1(3,3)$, $P_2(5,5)$
4. $P_1(0,3)$, $P_2(2,0)$
5. $P_1(-1,5)$, $P_2(3,-4)$

H.W 3- Find the terminal point of the vector $\overrightarrow{P_1P_2}=4i+8j$ if its initial point is (3, 10).

H.W 4- Find the initial point of the vector $\overrightarrow{P_1P_2}=\langle -5,-1\rangle$ if its terminal point is (4,7).

H.W 5- Determine a scalar k so that $\vec{A}=3i+kj$ and $\vec{B}=-i+9j$ are parallel.

H.W 6- Determine which of the following vectors are parallel to $\vec{A}=4i+6j$

1. $-4i-6j$
2. $8i+12j$
3. $-i-\frac{3}{2}j$
4. $2(i-j)-3\left(\frac{1}{2}i-\frac{5}{12}j\right)$
5. $(5i+j)-(7i+4j)$

H.W 7- find a vector **b** that is parallel to the given vector and has the indicated magnitude.

1. $\vec{A}=3i+7j$, $\|b\|=2$
2. $\vec{A}=\frac{1}{2}i-\frac{1}{2}j$, $\|b\|=3$

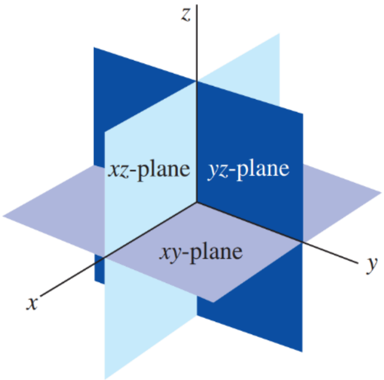
H.W 8- Find a vector in the opposite direction of $\vec{a}=\langle 4,10\rangle$ but $\frac{3}{4}$ as long.

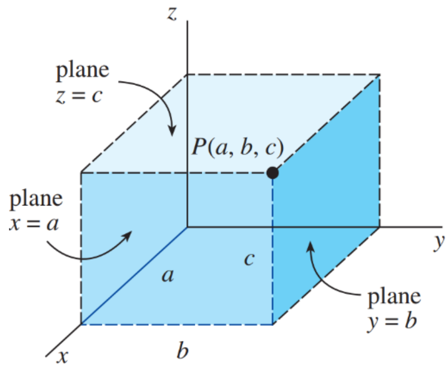
H.W 9- Given that $\vec{a}=\langle 1,1\rangle$ and $\vec{b}=\langle -1,0\rangle$. Find a vector in the same direction as but $\vec{a}+\vec{b}$ five times as long.

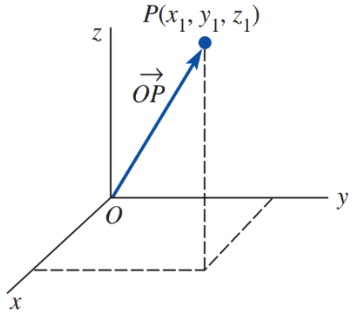
3- Vectors in 3-Space

A vector a in 3-space is any ordered triple of real numbers :

$$\vec{A} = \langle a_1, a_2, a_3 \rangle$$
$$\vec{A} = a_1 i + a_2 j + a_3 k$$

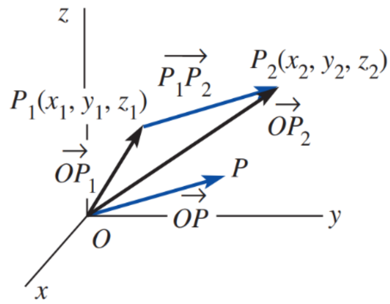






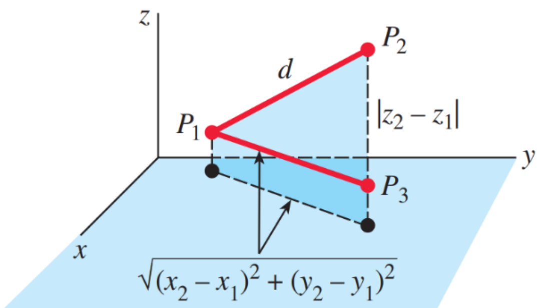
If $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ are the position vectors of the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, then the vector $\overrightarrow{P_1P_2}$ is given by :

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_1} - \overrightarrow{OP_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$$



➢ Distance Between Two Points

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



➢ Midpoint Formula

$$M(P_1, P_2) = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}, \frac{z_2 + z_1}{2} \right)$$

Definition

Addition	$\vec{A}+\vec{B}=(a_1+b_1)i+(a_2+b_2)j+(a_3+b_3)k$
Subtraction	$\vec{A}-\vec{B}=(a_1-b_1)i+(a_2-b_2)j+(a_3-b_3)k$
Scalar Multiplication	$c\vec{A}=ca_1i+ca_2j+ca_3k$
Negative	$-\vec{A}=-a_1i+(-a_2)j+(-a_3)k$
Zero vector	$0=\langle 0,0,0\rangle$
Magnitude	$\lVert\vec{A}\rVert=\sqrt{a_1^2+a_2^2+a_3^2}$
Equality	$\vec{A}=\vec{B}$ if and only if $a_1=b_1,\ a_2=b_2,\ a_3=b_3$
A Unit Vector	$\vec{u}=\frac{1}{\lVert\vec{A}\rVert}\vec{A}$ where $\lVert u\rVert=1$

Example [4] Find the vector $\overrightarrow{P_1P_2}$ if the points $P_1(4, 6,-2)$ and $P_2(1, 8, 3)$

Solution

$$\overrightarrow{P_1P_2}=\overrightarrow{OP_1}-\overrightarrow{OP_2}=\langle x_2-x_1,y_2-y_1,z_2-z_1\rangle=(x_2-x_1)i+(y_2-y_1)j+(z_2-z_1)k$$

$$\overrightarrow{P_1P_2}=(1-4)i+(8-6)j+(3-(-2))k=-3i+2j+5k$$

Example [5] Find a unit vector in the direction of $\vec{A}=\langle -2,3,6\rangle$.

Solution

$$\vec{u}=\frac{1}{\lVert\vec{A}\rVert}\vec{A} \qquad \lVert\vec{A}\rVert=\sqrt{a_1^2+a_2^2+a_3^2}=\sqrt{(-2)^2+3^2+6^2}=7$$

$$\vec{u}=\frac{1}{7}\langle -2,3,6\rangle=\left\langle -\frac{2}{7},\frac{3}{7},\frac{6}{7}\right\rangle=-\frac{2}{7}i+\frac{3}{7}j+\frac{6}{7}k$$

Example [6] If $\vec{A}=3i-4j+8k$ and $\vec{B}=i-4k$, find $5\vec{A}-2\vec{B}$.

Solution

$$\begin{aligned}5\vec{A}-2\vec{B}&=5(3i-4j+8k)-2(i-4k)\\&=(15i-20j+40k)-(2i+0j-8k)\\&=(15-2)i+(-20-0)j+(40-(-8))k\\&=13i-20j+48k\end{aligned}$$

Exercises 2-2

H.W 1- Find the vector $\overrightarrow{P_1P_2}$:

1. $P_1(3,4,5)$, $P_2(0,-2,6)$
2. $P_1(-2,4,0)$, $P_2\left(6,\frac{3}{2},8\right)$
3. $P_1(0,-1,0)$, $P_2(2,0,1)$
4. $P_1\left(\frac{1}{2},\frac{3}{4},5\right)$, $P_2\left(-\frac{5}{2},-\frac{9}{4},12\right)$

H.W 2- If $\vec{A}=i-3j+2k$, $\vec{B}=-i+j+k$, and $\vec{C}=2i+6j+9k$. Find the vector or scalar.

1. $\vec{A}+\vec{B}+\vec{C}$
2. $2\vec{A}-(\vec{B}-\vec{C})$
3. $\vec{B}+2(\vec{A}-3\vec{C})$
4. $\|\vec{C}\|\|2\vec{B}\|$
5. $\|\vec{B}\|\vec{A}+\|\vec{A}\|\vec{B}$
6. $\left\|\frac{\vec{A}}{\|\vec{A}\|}\right\|+5\left\|\frac{\vec{B}}{\|\vec{B}\|}\right\|$

*H.W 3-*Find a vector $\vec{B}$ that is four times as long as $\vec{A}=i-j+k$ in the same direction.

*H.W 4-*Find a vector $\vec{B}$ for which $\|\vec{B}\|=\frac{1}{2}$ that is parallel to $\vec{A}=-6i+3j-2k$ but has the opposite direction.

*H.W 5-*solve for the unknown

1. $P_1(x,2,3)$, $P_2(2,1,1)$, $d(P_1,P_2)=\sqrt{21}$
2. $P_1(x,x,1)$, $P_2(0,3,5)$, $d(P_1,P_2)=5$

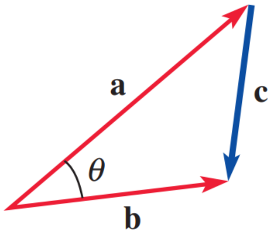
4- The Dot Product

A product of two vectors $\vec{A}$ and $\vec{B}$ can be formed in such a way that the result is a scalar. The result is written $\vec{A}\cdot\vec{B}$ and called the dot product of **a** and **b**. The names scalar product and inner product are also used in place of the term *dot product*.

$$\vec{A}\cdot\vec{B}=a_1b_1+a_2b_2+a_3b_3$$

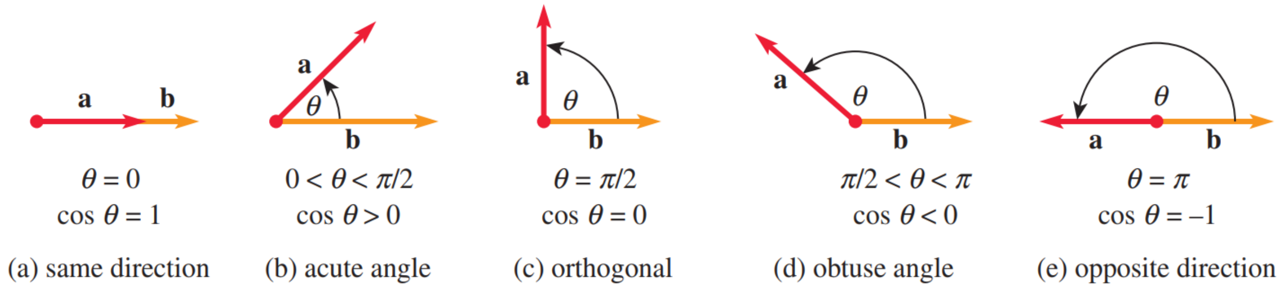
Let $\vec{A}$ and $\vec{B}$ be any two vectors that after a translation to bring their bases into coincidence are inclined to one another at an angle θ:

$$\vec{A}\cdot\vec{B}=\|\vec{A}\|\|\vec{B}\|\cos\theta$$



Orthogonal Vectors

Two nonzero vectors $\vec{A}$ and $\vec{B}$ are orthogonal if and only if $\vec{A}\cdot\vec{B}=0$.



Example [7] If $\vec{A}=10i+2j-6k$ and $\vec{B}=-\frac{1}{2}i+4j-3k$, find $\vec{A}\cdot\vec{B}$ and angle θ.

Solution

$$\vec{A}\cdot\vec{B}=a_1b_1+a_2b_2+a_3b_3=10\times\left(-\frac{1}{2}\right)+2\times4+(-6)\times(-3)=21$$

$$\|\vec{A}\|=\sqrt{a_1^2+a_2^2+a_3^2}=\sqrt{10^2+2^2+(-6)^2}=\sqrt{140}$$

$$\|\vec{B}\|=\sqrt{b_1^2+b_2^2+b_3^2}=\sqrt{\left(-\frac{1}{2}\right)^2+4^2+(-3)^2}=\frac{1}{2}\sqrt{101}$$

$$\vec{A}\cdot\vec{B}=\|\vec{A}\|\|\vec{B}\|\cos\theta$$

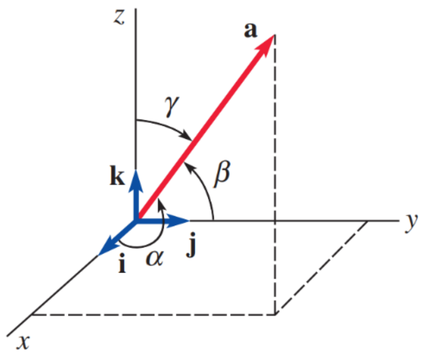
$$21=\sqrt{140}\times\frac{1}{2}\sqrt{101}\cos\theta$$

$$\theta=\cos^{-1}(0.3532)=69.32^o$$

Direction Cosines

For a nonzero vector $\vec{A} = a_1 i + a_2 j + a_3 k$ in 3-space, the angles α, β, and γ between $\vec{A}$ and the unit vectors i, j, and k, respectively, are called direction angles of $\vec{A}$.

$$\cos\alpha = \frac{a_1}{\|\vec{A}\|} \quad , \quad \cos\beta = \frac{a_2}{\|\vec{A}\|} \quad , \quad \cos\gamma = \frac{a_3}{\|\vec{A}\|}$$



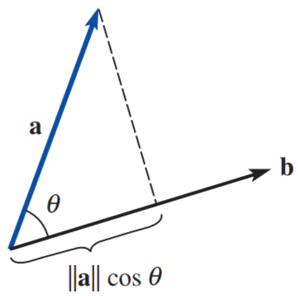
Component of A on B

we write the components of $\vec{A}$ as :

$$\text{comp}_i \vec{A} = \vec{A}\cdot i \quad , \quad \text{comp}_j \vec{A} = \vec{A}\cdot j \quad , \quad \text{comp}_k \vec{A} = \vec{A}\cdot k$$

The component of $\vec{A}$ on an arbitrary vector $\vec{B}$.

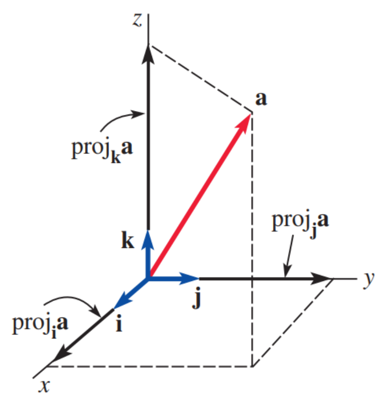
$$\text{comp}_{\vec{B}} \vec{A} = \frac{\vec{A}\cdot\vec{B}}{\|\vec{B}\|}$$



Projection of A onto B

The projection of a vector $\vec{A}$ in any of the directions determined by i, j, k is simply the vector formed by multiplying the component of $\vec{A}$ in the specified direction with a unit vector in that direction.

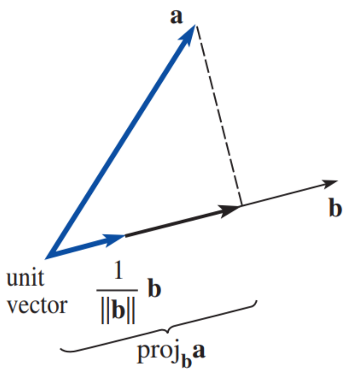
$$\text{proj}_i \vec{A} = \left(\text{comp}_i \vec{A}\right) i = \left(\vec{A}\cdot i\right) i = a_1 i$$



The general case of the projection of $\vec{A}$ onto $\vec{B}$

$$\text{proj}_{\vec{B}} \vec{A} = \left(\text{comp}_{\vec{B}} \vec{A}\right)\left(\frac{1}{\|\vec{B}\|}\vec{B}\right)$$

$$\text{proj}_{\vec{B}} \vec{A} = \left(\frac{\vec{A}\cdot\vec{B}}{\|\vec{B}\|^2}\right)\vec{B}$$



Example [8] Find the direction cosines and direction angles of the vector $\vec{A} = 2i + 5j + 4k$

Solution

$$\|\vec{A}\| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{2^2 + 5^2 + 4^2} = 3\sqrt{5}$$

$$\cos\alpha = \frac{a_1}{\|\vec{A}\|} = \frac{2}{3\sqrt{5}} \qquad \alpha = \cos^{-1}\left(\frac{2}{3\sqrt{5}}\right) = 72.65^o$$

$$\cos\beta = \frac{a_2}{\|\vec{A}\|} = \frac{5}{3\sqrt{5}} \qquad \beta = \cos^{-1}\left(\frac{5}{3\sqrt{5}}\right) = 41.81^o$$

$$\cos\gamma = \frac{a_3}{\|\vec{A}\|} = \frac{4}{3\sqrt{5}} \qquad \gamma = \cos^{-1}\left(\frac{4}{3\sqrt{5}}\right) = 53.39^o$$

$$Check \qquad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{2}{3\sqrt{5}}\right)^2 + \left(\frac{5}{3\sqrt{5}}\right)^2 + \left(\frac{4}{3\sqrt{5}}\right)^2 = 1 \quad O.K$$

Example [9] Let $\vec{A} = 2i + 3j - 4k$ and $\vec{B} = i + j + 2k$ .Find $\text{comp}_{\vec{B}}\vec{A}$ and $\text{comp}_{\vec{A}}\vec{B}$.

Solution

$$\|\vec{A}\| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{2^2 + 3^2 + (-4)^2} = \sqrt{29}$$

$$\|\vec{B}\| = \sqrt{b_1^2 + b_2^2 + b_3^2} = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$$

$$\vec{A}\cdot\vec{B} = a_1b_1 + a_2b_2 + a_3b_3 = 2\times1 + 3\times1 + (-4)\times2 = -3$$

$$\text{comp}_{\vec{B}}\vec{A} = \frac{\vec{A}\cdot\vec{B}}{\|\vec{B}\|} = \frac{-3}{\sqrt{6}}$$

$$\text{comp}_{\vec{A}}\vec{B} = \frac{\vec{A}\cdot\vec{B}}{\|\vec{A}\|} = \frac{-3}{\sqrt{29}}$$

Example [10] Let $\vec{A} = 4i + j$ and $\vec{B} = 2i + 3j$ .Find $\text{proj}_{\vec{B}}\vec{A}$ and $\text{proj}_{\vec{A}}\vec{B}$.

Solution

$$\|\vec{A}\| = \sqrt{a_1^2 + a_2^2} = \sqrt{4^2 + 1^2} = \sqrt{17}$$

$$\|\vec{B}\| = \sqrt{b_1^2 + b_2^2} = \sqrt{2^2 + 3^2} = \sqrt{13}$$

$$\vec{A}\cdot\vec{B} = a_1b_1 + a_2b_2 = 4\times2 + 1\times3 = 11$$

$$\text{proj}_{\vec{B}}\vec{A} = \left(\frac{\vec{A}\cdot\vec{B}}{\|\vec{B}\|^2}\right)\vec{B} = \frac{11}{13}(2i + 3j) = \frac{22}{13}i + \frac{33}{13}j$$

$$\text{proj}_{\vec{A}}\vec{B} = \left(\frac{\vec{A}\cdot\vec{B}}{\|\vec{A}\|^2}\right)\vec{A} = \frac{11}{17}(4i + j) = \frac{44}{17}i + \frac{11}{17}j$$

➢ Exercises 2-3

H.W 1- If $\vec{A} = 2i - 3j + 4k$, $\vec{B} = -i + 2j + 5k$ and $\vec{C} = 3i + 6j - k$, Find the indicated scalar or vector :

1. $\vec{A} \cdot \vec{B}$
2. $(2\vec{A}) \cdot (\vec{A} - 2\vec{B})$
3. $\vec{A} \cdot \vec{C}$
4. $(\vec{C} \cdot \vec{B})\vec{A}$
5. $\vec{A} \cdot (\vec{A} + \vec{B} + \vec{C})$
6. $(\vec{C} \cdot \vec{B})(\vec{A} \cdot \vec{C})$
7. $\vec{B} \cdot (\vec{A} - \vec{C})$
8. $\left(\dfrac{\vec{A} \cdot \vec{B}}{\vec{B} \cdot \vec{B}}\right)\vec{B}$
9. $(2\vec{B}) \cdot (3\vec{C})$
10. $\left(\dfrac{\vec{A} \cdot \vec{C}}{\vec{C} \cdot \vec{B}}\right)\vec{C}$

H.W 2- Determine a scalar x so that the given vectors are orthogonal.

1. $\vec{A} = 2i - xj + 3k$, $\vec{B} = 3i + 2j + 4k$
2. $\vec{A} = xi + \frac{1}{2}j + xk$, $\vec{B} = -3i + 4j + xk$

H.W 3- Find a vector $\vec{V} = xi + yj + k$ that is orthogonal to both $\vec{A} = 3i + j + k$ and $\vec{B} = 3i + 2j + 2k$.

H.W 4- Determine a scalar c so that the angle between $\vec{A} = i + cj$ and $\vec{B} = i + j$ is 45°.

H.W 5- If $\vec{A} = i - j + 3k$ and $\vec{B} = 2i + 6j + 3k$, Find the vector $\text{proj}_{(\vec{A}+\vec{B})}\vec{A}$ and $\text{proj}_{(\vec{A}-\vec{B})}\vec{B}$.

H.W 6- If $\vec{A} = 4i + 3j$ and $\vec{B} = -i + j$, Find the vector $\text{comp}_{\vec{A}}(\vec{B} - \vec{A})$ and $\text{comp}_{2\vec{B}}(\vec{A} + \vec{B})$.

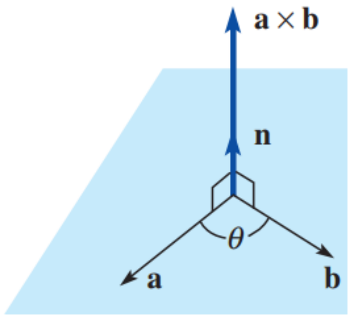
5- Cross Product

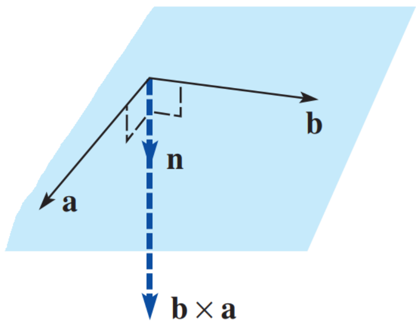
The cross product, introduced in this section, is only defined for vectors in ***3-space*** and results in another vector in 3-space. $\vec{A}\times\vec{B}$ is ***orthogonal*** to the plane containing $\vec{A}$ and $\vec{B}$. The cross product of two vectors $\vec{A}$ and $\vec{B}$ is the vector :

$$\vec{N}=\vec{A}\times\vec{B}=\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$=\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i-\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j+\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

$$=(a_2b_3-a_3b_2)i-(a_1b_3-a_3b_1)j+(a_1b_2-a_2b_1)k$$





Magnitude of the Cross Product

For nonzero vectors $\vec{A}$ and $\vec{B}$, if θ is the angle between $\vec{A}$ and $\vec{B}$ ($0\le\theta\le\pi$), then

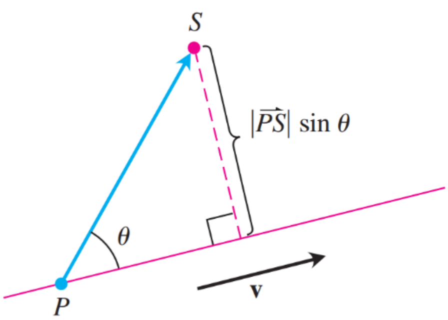
$$\|\vec{A}\times\vec{B}\|=\|\vec{A}\|\|\vec{B}\|\sin\theta$$

Parallel Vectors

Two nonzero vectors $\vec{A}$ and $\vec{B}$ are parallel if and only if $\|\vec{A}\times\vec{B}\|=0$.

The Distance from a Point to a Line in Space

$$D=\frac{\|\overrightarrow{PS}\times\vec{V}\|}{\|\vec{V}\|}$$



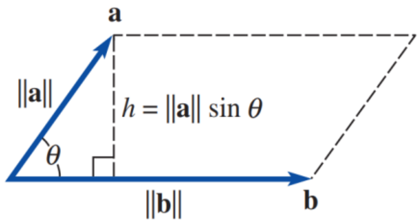
Areas

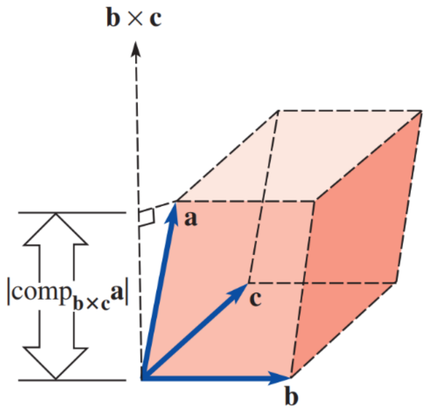
Two nonzero and nonparallel vectors $\vec{A}$ and $\vec{B}$ can be considered to be the sides of a parallelogram. The area A of a *parallelogram* is :

$$Area = base \times height = \|\vec{A} \times \vec{B}\|$$

the area of a *triangle* is :

$$Area = \frac{1}{2}\|\vec{A} \times \vec{B}\|$$





Volume of a Parallelepiped

If the vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ do not lie in the same plane, then the volume of the parallelepiped with edges a, b, and c shown in figure is the absolute value of the scalar triple product of the vectors.:

$$\begin{aligned}\text{Volume} &= (\text{area of base})\times(\text{height})\\ &= \|\vec{B}\times\vec{C}\|\left|\text{comp}_{\vec{B}\times\vec{C}}\vec{A}\right|\\ &= \|\vec{B}\times\vec{C}\|\left|\vec{A}\cdot\left(\frac{1}{\|\vec{B}\times\vec{C}\|}\vec{B}\times\vec{C}\right)\right|\\ &= \left|\vec{A}.(\vec{B}\times\vec{C})\right|\\ &= \begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix} \qquad \text{absolute}\end{aligned}$$

Example [11] Let $\vec{A} = 4i - 2j + 5k$ and $\vec{B} = 3i + j - k$ .Find $\vec{A}\times\vec{B}$.

Solution

$$\vec{A}\times\vec{B} = \begin{vmatrix} i & j & k\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

$$= \begin{vmatrix} i & j & k\\ 4 & -2 & 5\\ 3 & 1 & -1\end{vmatrix} = \begin{vmatrix} -2 & 5\\ 1 & -1\end{vmatrix} i - \begin{vmatrix} 4 & 5\\ 3 & -1\end{vmatrix} j + \begin{vmatrix} 4 & -2\\ 3 & 1\end{vmatrix} k$$

$$= ((-2)(-1) - 5\times1)i - (4\times(-1) - 5\times3)j + (4\times1 - (-2)\times3)k$$

$$= -3i + 19j + 10k$$

Example [12] Determine whether $\vec{A}=2i+j-k$ and $\vec{B}=-6i-3j+3k$ are parallel vectors.

Solution

$$\vec{A}\times\vec{B}=\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$=\begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ -6 & -3 & 3 \end{vmatrix}=\begin{vmatrix} 1 & -1 \\ -3 & 3 \end{vmatrix}i-\begin{vmatrix} 2 & -1 \\ -6 & 3 \end{vmatrix}j+\begin{vmatrix} 2 & 1 \\ -6 & -3 \end{vmatrix}k$$

$$=(1\times3-(-3)\times(-1))i-(2\times3-(-6)\times(-1))j+(2\times(-3)-(-6)\times1)k$$

$$=0i+0j+0k=0$$

The vectors $\vec{A}$ and $\vec{B}$ are parallel vectors.

Example [13] Find the area of the triangle determined by the points $P_1(1,1,1)$, $P_2(2,3,4)$, and $P_3(3,0,-1)$.

Solution

$$\overrightarrow{P_1P_2}=(2-1)i+(3-1)j+(4-1)k=i+2j+3k$$

$$\overrightarrow{P_1P_3}=(3-1)i+(0-1)j+(-1-1)k=2i-j-2k$$

$$\vec{V}=\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}=\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$=\begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 2 & -1 & -2 \end{vmatrix}=\begin{vmatrix} 2 & 3 \\ -1 & -2 \end{vmatrix}i-\begin{vmatrix} 1 & 3 \\ 2 & -2 \end{vmatrix}j+\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}k$$

$$=(2\times(-2)-(-1)\times3)i-(1\times(-2)-2\times3)j+(1\times(-1)-2\times2)k$$

$$=-i+8j-5k$$

$$\text{Area}=\frac{1}{2}\|\vec{V}\|=\sqrt{v_1^2+v_2^2+v_3^2}=\frac{1}{2}\sqrt{(-1)^2+8^2+(-5)^2}=\frac{3}{2}\sqrt{10}\ \ unit^2$$

Example [14] Find the volume of the parallelepiped for the vectors are three edges. $\vec{A}=3i+j-2k$, $\vec{B}=-2i+4j+3k$ and $\vec{C}=-i+5j-2k$.

Solution

$$\text{Volume}=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=\begin{vmatrix} 3 & 1 & -2 \\ -2 & 4 & 3 \\ -1 & 5 & -2 \end{vmatrix}=3\times\begin{vmatrix} 4 & 3 \\ 5 & -2 \end{vmatrix}-1\times\begin{vmatrix} -2 & 3 \\ -1 & -2 \end{vmatrix}+(-2)\times\begin{vmatrix} -2 & 4 \\ -1 & 5 \end{vmatrix}$$

$$=3\times(4\times(-2)-5\times3)-1\times((-2)\times(-2)-(-1)\times3)+(-2)\times((-2)\times5-(-1)\times4)$$

$$=3\times(-23)-1\times(7)+(-2)\times(-6)=-69-7+12=-78+12=-66$$

$$\text{Volume}=66\quad unit^3\quad \text{absolute}$$

➢ Exercises 2-4

H.W 1- Find $\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}$, Find Distance ($\overrightarrow{P_1P_2}, P_3$) and Distance ($\overrightarrow{P_1P_3}, P_2$):

H.W 1- $P_1(2, 1, 3)$, $P_2(0, 3, 1)$, $P_3(1, 2, 4)$

H.W 2- $P_1(0, 0, 1)$, $P_2(0, 1, 2)$, $P_3(1, 2, 3)$

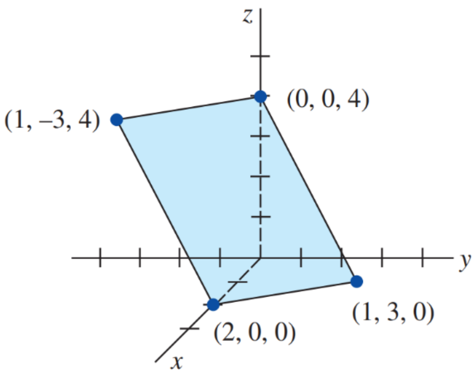
H.W 2- Find a vector that is perpendicular to both $\vec{A}$ and $\vec{B}$

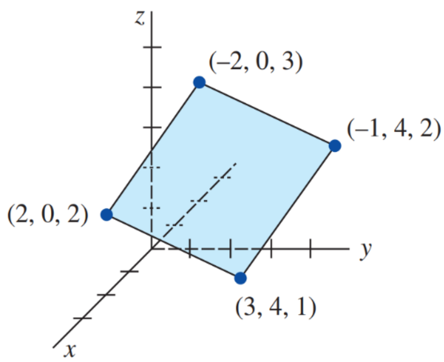
1. $\vec{A} = 2i + 7j - 4k$, $\vec{B} = i + j - k$
2. $\vec{A} = -i - 2j + 4k$, $\vec{B} = 4i - j$

H.W 3- If $\vec{A} = 3i + j - 2k$, $\vec{B} = -2i + 4j + 3k$ and $\vec{C} = -i + 5j - 2k$. Find the indicated scalar or vector.

1. $\vec{A}\times(3\vec{B})$
2. $(\vec{A}\times\vec{B})\times\vec{C}$
3. $(\vec{A}\times\vec{B})\cdot\vec{C}$
4. $\vec{A}\times\vec{B}+\vec{A}\times\vec{C}+\vec{B}\times\vec{C}$
5. $(4\vec{A})\cdot(\vec{B}\times\vec{C})$

H.W 4- (a) verify that the given quadrilateral is a parallelogram, and (b) find the area of the parallelogram.





H.W 5- Find the area of the triangle determined by the given points.

1. $P_1(1, 1, 1)$, $P_2(1, 2, 1)$, $P_3(1, 1, 2)$
2. $P_1(0, 0, 0)$, $P_2(0, 1, 2)$, $P_3(2, 2, 0)$
3. $P_1(1, 2, 4)$, $P_2(1, 1, 3)$, $P_3(1, 1, 2)$
4. $P_1(1, 0, 3)$, $P_2(0, 0, 6)$, $P_3(2, 4, 5)$

H.W 6- Find the volume of the parallelepiped for which the given vectors are three edges.

1. $\vec{A} = i + j$, $\vec{B} = -i + 4j$, $\vec{C} = 2i + 2j + 2k$
2. $\vec{A} = 3i + j + k$, $\vec{B} = i + 4j + k$, $\vec{C} = i + j + 5k$

H.W 7- Determine whether the four points $P_1(1, 1, 2)$, $P_2(4, 0, 3)$, $P_3(1, 5, 10)$, and $P_4(7, 2, 4)$ lie in the same plane.

6- Lines and Planes in 3-Space

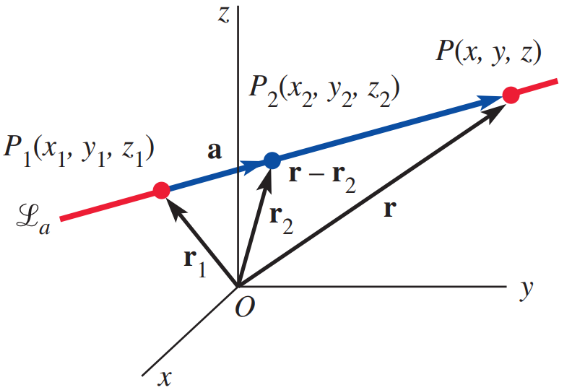
As in the plane, any two distinct points in 3-space determine only one line between them. To find an equation of the line through $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, let us assume that $P(x, y, z)$ is *any* point on the line. In figure, if $\mathbf{r}=\overrightarrow{OP}$, $\mathbf{r}_1=\overrightarrow{OP_1}$, and $\mathbf{r}_2=\overrightarrow{OP_2}$, we see that vector $a=r_2-r_1$ is parallel to vector $r-r_2$.

➢ A vector equation

A vector equation for the line is

$$r=r_2+at$$
$$a=r_2-r_1=\langle x_2-x_1, y_2-y_1, z_2-z_1\rangle=\langle a_1,a_2,a_3\rangle$$

The scalar t is called a ***parameter*** and the nonzero vector $\vec{A}$ is called a ***direction vector***.



➢ Parametric Equations

$$x=x_2+a_1t$$
$$y=y_2+a_2t$$
$$z=z_2+a_3t$$

Example [15] Find a vector equation for the line through (2,-1,8) and (5,6,-3). Then find parametric equations for the line.

Solution

$P_1(x_1,y_1,z_1)=(2,-1,8)$

$P_2(x_2,y_2,z_2)=(5,6,-3)$

$\vec{a}=\overrightarrow{P_1P_2}=(5-2)i+(6-(-1))j+(-3-8)k=3i+7j-11k=\langle 3,7,-11\rangle$

A vector equation

$\langle x,y,z\rangle=\langle 5,6,-3\rangle+\langle 3,7,-11\rangle t$

The parametric equations

$x=x_2+a_1t \quad\Rightarrow\quad x=5+3t$

$y=y_2+a_2t \quad\Rightarrow\quad y=6+7t$

$z=z_2+a_3t \quad\Rightarrow\quad z=-3+11t$

Example [16] Find a vector that is parallel to the line La whose parametric equations are $x = 4 + 9t$, $y = -14 + 5t$, $z = 1 - 3t$.

Solution

$\vec{A} = 9i + 5j - 3k$

Example [17] Write vector, and parametric equations for the line through (4, 6,-3) and parallel to $\vec{A} = 5i - 10j + 2k$.

Solution

A vector equation

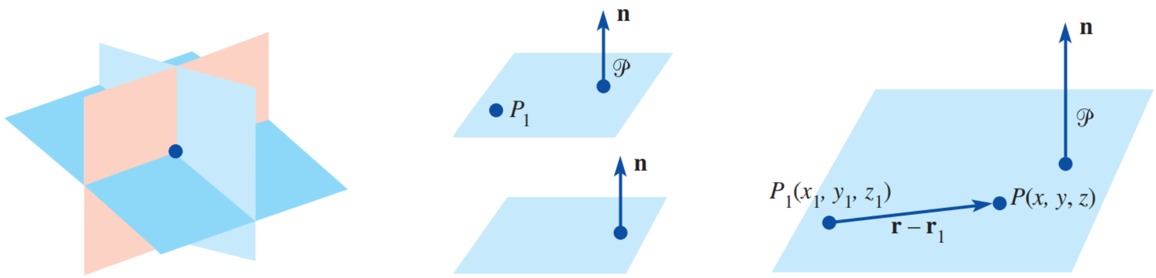
$\langle x, y, z \rangle = \langle 4, 6, -3 \rangle + \langle 5, -10, 2 \rangle t$

The parametric equations

$$x = x_2 + a_1 t \quad \Rightarrow \quad x = 4 + 5t$$
$$y = y_2 + a_2 t \quad \Rightarrow \quad y = 6 - 10t$$
$$z = z_2 + a_3 t \quad \Rightarrow \quad z = -3 + 2t$$

➢ Planes: Vector Equation

$$\vec{N}.(r - r_1) = 0$$



➢ Cartesian Equation

If the normal vector is $\vec{N} = Ai + Bj + Ck$, then yields a Cartesian equation of the plane containing $P_o(x_o, y_o, z_o)$:

$$A(x - x_o) + B(y - y_o) + C(z - z_o) = 0$$
$$Ax + By + Cz = Ax_o + By_o + Cz_o$$

Example [18] Find an equation of the plane with normal vector $\vec{N} = 2i + 8j - 5k$ containing the point (4,-1,3).

Solution

$$A(x - x_o) + B(y - y_o) + C(z - z_o) = 0$$
$$Ax + By + Cz = Ax_o + By_o + Cz_o$$
$$2x + 8y - 5z = 2 \times (4) + 8 \times (-1) + (-5) \times 3$$
$$2x + 8y - 5z = -15$$

Example [19] Find an equation of the plane that contains (1,0,-1), (3,1,4), and (2,-2,0).

Solution

$$\vec{A} = (3-1)i + (1-0)j + (4-(-1))k = 2i + j + 5k$$

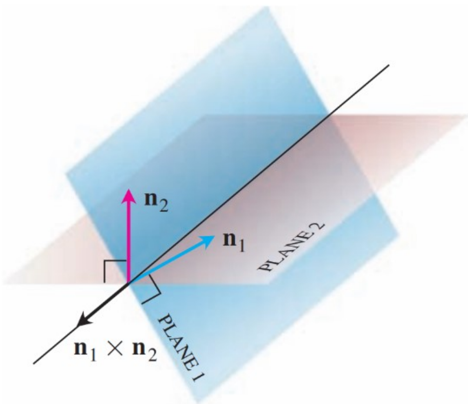
$$\vec{B} = (2-1)i + (-2-0)j + (0-(-1))k = i - 2j + k$$

$$\vec{N} = \vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 2 & 1 & 5 \\ 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 5 \\ -2 & 1 \end{vmatrix} i - \begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} j + \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} k = 11i + 3j - 5k$$

$$11x + 3y - 5z = 11\times 1 + 3\times 0 - 5\times(-1)$$

$$11x + 3y - 5z = 16$$

➢ Line of Intersection of Two Planes



Example [20] Find parametric equations for the line of intersection of $2x - 3y + 4z = 1$, $x - y - z = 5$

Solution

$$\vec{N_1} = 2i - 3j + 4k$$

$$\vec{N_2} = i - j - k$$

$$\vec{N} = \vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 2 & -3 & 4 \\ 1 & -1 & -1 \end{vmatrix} = \begin{vmatrix} -3 & 4 \\ -1 & -1 \end{vmatrix} i - \begin{vmatrix} 2 & 4 \\ 1 & -1 \end{vmatrix} j + \begin{vmatrix} 2 & -3 \\ 1 & -1 \end{vmatrix} k = 7i + 6j + k$$

A intersection point $P_o(x_o, y_o, z_o)$ take $z_o = 0$

$$2x_o - 3y_o + 4(0) = 1 \quad \Rightarrow \quad 2x_o - 3y_o = 1 \quad ---[1]$$

$$x_o - y_o - 0 = 5 \quad \Rightarrow \quad x_o - y_o = 5 \quad ---[2]$$

Solve $P_o(14,9,0)$

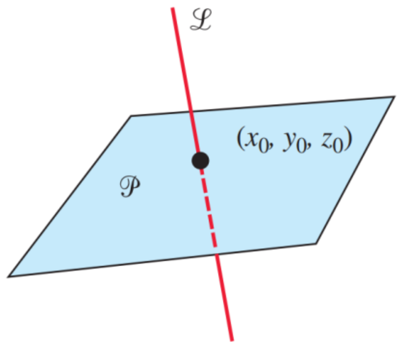
The parametric equations

$$x = x_o + a_1 t \quad \Rightarrow \quad x = 14 + 7t$$

$$y = y_o + a_2 t \quad \Rightarrow \quad y = 9 + 6t$$

$$z = z_o + a_3 t \quad \Rightarrow \quad z = -t$$

➢ Point of Intersection of a Line and a Plane



Example [21] Find the point of intersection of the plane $3x-2y+z=-5$ and the line $x=1+t, y=-2+2t, z=4t$.

Solution

Substituting the parametric equations into the equation of the plane gives :

$$x=1+t \qquad y=-2+2t \qquad z=4t$$

$$3(1+t)-2(-2+2t)+(4t)=-5 \quad \Rightarrow \quad t=-4$$

$$\left.\begin{aligned} x_i &= 1+(-4)=-3 \\ y_i &= -2+2(-4)=-10 \\ z_i &= 4(-4)=-16 \end{aligned}\right\} \Rightarrow \quad P(-3,-10,-16)$$

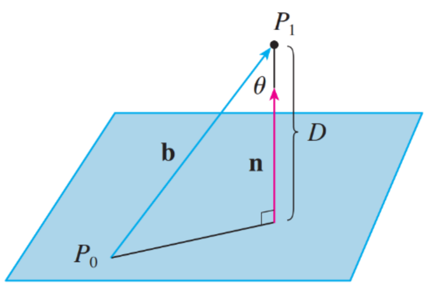
➢ The Distance from a Point to a Plane

A point $\quad P_1(x_1, y_1, z_1)$

The y-intercept point $\quad P_o(0, y_o, 0)$

The plane $\quad Ax+By+Cz=d$

$$D=\frac{\left|\overrightarrow{P_oP_1}\cdot\overrightarrow{N}\right|}{\left\|\overrightarrow{N}\right\|}$$



Example [22] Find the distance from $P(1,1,3)$ to the plane $3x+2y+6z=6$.

Solution

A point $\quad P_1(1,1,3)$

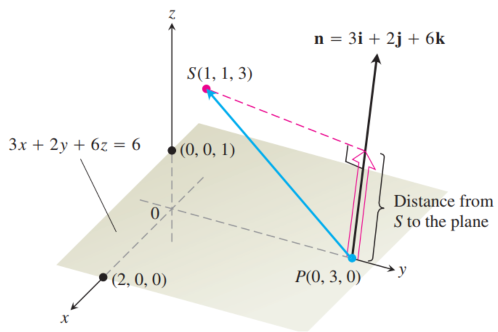
The y-intercept point $\quad P_o(0, y_o, 0) \quad 3(x)+2y_o+6(z)=6 \qquad y_o=3 \qquad P_o(0,3,0)$

$\overrightarrow{P_oP_1}=i-2j+3k \qquad \overrightarrow{N}=3i+2j+6k$

$\overrightarrow{N}\cdot\overrightarrow{B}=1\times3-2\times2+3\times6=17$

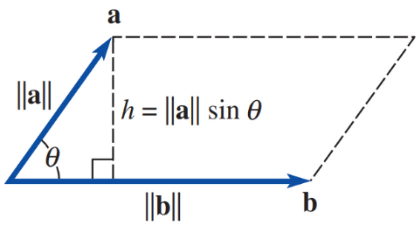
$\left\|\overrightarrow{N}\right\|=\sqrt{3^2+2^2+6^2}=7$

$$D=\frac{\left|\overrightarrow{P_oP_1}\cdot\overrightarrow{N}\right|}{\left\|\overrightarrow{N}\right\|}=\frac{17}{7}$$



Example [23] Find the distance from the point S(1,1,5) to the line x = 1+t, y =3-t, z =2t.

Solution



$$\vec{V} = i - j + 2t$$

$$P(1,3,0)$$

$$S(1,1,5)$$

$$\overrightarrow{PS} = (1-1)i + (1-3)j + (5-0)k = -2i + 5k$$

$$\overrightarrow{PS} \times \vec{V} = \begin{vmatrix} i & j & k \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} -2 & 5 \\ -1 & 2 \end{vmatrix} i - \begin{vmatrix} 0 & 5 \\ 1 & 2 \end{vmatrix} j + \begin{vmatrix} 0 & -2 \\ 1 & -1 \end{vmatrix} k = i + 5j + 2k$$

$$D = \frac{\|\overrightarrow{PS} \times \vec{V}\|}{\|\vec{V}\|} = \frac{\sqrt{1^2 + 5^2 + 2^2}}{\sqrt{1^2 + (-1)^2 + 2^2}} = \sqrt{5}$$

Example [24] Find the distance between the parallel planes 10x+2y-2z =5 and 5x+y-z =1.

Solution

A Plane (1) $10x + 2y - 2z = 5$

A Plane (2) $5x + y - z = 1$

The y-intercept a point in a plane (1) $P_o(0, y_o, 0)$

$$10 \times 0 + 2y_o - 2 \times 0 = 5 \qquad y_o = \frac{5}{2} \qquad P_o\left(0, \frac{5}{2}, 0\right)$$

The y-intercept a point in a plane (2) $P_1(0, y_o, 0)$

$$5(0) + y_o - 0 = 1 \qquad y_o = 1 \qquad P_1(0,1,0)$$

$$\overrightarrow{P_oP_1} = -\frac{3}{2}j \qquad \vec{N} = 10i + 2j - 2k$$

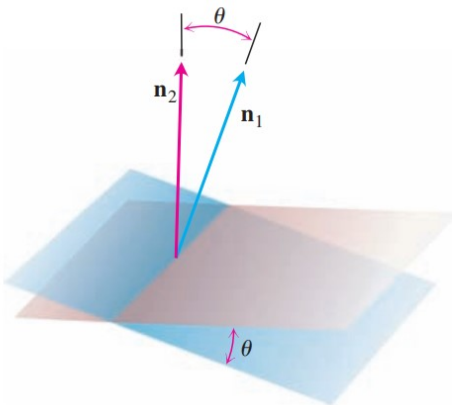
$$\overrightarrow{P_oP_1} \cdot \vec{N} = -3$$

$$\|\vec{N}\| = \sqrt{10^2 + 2^2 + (-2)^2} = 6\sqrt{3}$$

$$D = \frac{|\overrightarrow{P_oP_1} \cdot \vec{N}|}{\|\vec{N}\|} = \frac{3}{6\sqrt{3}} = \frac{1}{2\sqrt{3}}$$

Angles Between Planes

$$\cos\theta = \frac{\vec{N_1} \cdot \vec{N_2}}{\|\vec{N_1}\| \|\vec{N_2}\|}$$



➢ Exercises 2-5

H.W 1- Find a vector equation and the parametric equations for the line through the given points.

1) – $(2,3,5),(6,-1,8)$
2) – $(1,0,0),(3,-2,-7)$
3) – $\left(4,\frac{1}{2},\frac{1}{3}\right),\left(-6,-\frac{1}{4},\frac{1}{6}\right)$
4) – $(2,0,0),(0,4,9)$
5) – $(0,0,5),(-2,4,0)$
6) – $(-3,7,9),(4,-8,-1)$

H.W 2- Determine the points of intersection of the given line and the three coordinate planes.

1. $x=4-2t, y=1+2t, z=9+3t$.
2. $x=1+2t, y=-2+3t, z=4+2t$

H.W 3- Determine whether the given lines intersect. If so, find the point of intersection.

1. $x=4+t, y=5+t, z=-1+2t$
   $x=6+2s, y=11+4s, z=-3+s$
2. $x=1+t, y=2-t, z=3t$
   $x=2-s, y=1+s, z=6s$
3. $x=2-t, y=3+t, z=1+t$
   $x=4+s, y=1+s, z=1-s$
4. $x=3-t, y=2+t, z=8+2t$
   $x=2+2s, y=-2+3s, z=-2+8s$

H.W 4- Find an equation of the plane that satisfies the given conditions.

1. Contains (2,3,-5) and is parallel to x+y-4z=1.
2. Contains the origin and is parallel to 5x-y+z=6.
3. Contains (3, 6, 12) and is parallel to the xy-plane.
4. Contains (7, 5, 18) and is perpendicular to the y-axis.
5. Contains the lines x=1+3t, y=1-t, z=2+t; x=4+4s, y=2s, z=3+s.

H.W 5- The line through (2,3,0) perpendicular to the vectors u=i+2j+3k and v= 3i+4j+5k

H.W 6- Find the distance from the point to the given line x =1+t, y= 3-2 2t, z=4- 3t and point (4,1,-2)

H.W 7- Find the distance from the point to the given plane. (1, -2, 4), 3x + 2y+ 6z= 5

H.W 8- Find equations of the planes that are parallel to the plane x +2y - 2z= 1 and two units away from it

Chapter Three

Partial Derivatives

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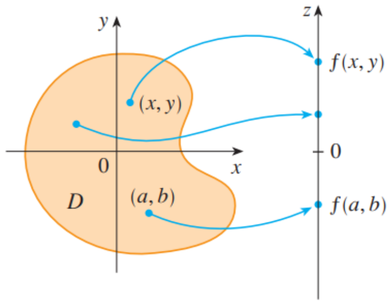
Civil Engineering Department

1- Functions of Several Variables

Suppose D is a set of n-tuples of real numbers $(x_1, x_2, x_3, \ldots, x_n)$ a real-valued function f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, x_3, \ldots, x_n)$$

to each element in D. The set D is the function's domain. The set of w-values taken on by f is the function's range. The symbol w is the dependent variable of f, and f is said to be a function of the n independent variables to We also call the 's the function's input variables and call w the function's output variable.



➢ Domains and Ranges

Example [1] For each of the following functions, find the domain and range.

$(a)\quad f(x,y) = \sqrt{y - x^2}$ $\qquad (b)\quad f(x,y) = x\ln(y^2 - x)$

$(c)\quad f(x,y) = \sqrt{9 - x^2 - y^2}$ $\qquad (d)\quad f(x,y) = \dfrac{\sqrt{x+y+1}}{x-1}$

Solution

$(a)\quad f(x,y) = \sqrt{y - x^2}$

Domain

$y - x^2 \geq 0$

$x^2 \leq y$

$\therefore D = \{(x,y) \mid y - x^2 \geq 0, x^2 \leq y\}$

Range $\qquad \therefore \mathrm{R} = [0, +\infty)$

$(b)\quad f(x,y) = x\ln(y^2 - x)$

Domain

$y^2 - x > 0$

$x < y^2$

$\therefore D = \{(x,y) \mid y^2 - x > 0, x < y^2\}$

Range $\qquad \therefore \mathrm{R} = (-\infty, +\infty)$

$(b)\quad f(x,y)=\sqrt{9-x^2-y^2}$

Domain

$9-x^2-y^2\geq 0$

$x^2+y^2\leq 9$

$\therefore D=\{(x,y)|x^2+y^2\leq 9\}$

Range $\quad \therefore R=(0,+\infty]$

$(d)\quad f(x,y)=\dfrac{\sqrt{x+y+1}}{x-1}$

Domain

$x+y+1\geq 0 \qquad x+y\geq -1$

$x\neq 1$

$\therefore D=\{(x,y)|x+y\geq -1, x\neq 1\}$

Range $\quad \therefore R=(-\infty,0)\cup(0,+\infty)$

➢ *Exercises 3-1*

H.W 1- Find the domain for each function :

(1) $f(x,y)=\sqrt{y-x-2}$

(2) $f(x,y)=\ln(x^2+y^2-4)$

(3) $f(x,y)=\dfrac{(x-1)(y+2)}{(y-x)(y-x^3)}$

(4) $f(x,y)=\dfrac{\sin xy}{x^2+y^2-25}$

(5) $f(x,y)=\cos^{-1}(y-x^2)$

(6) $f(x,y)=\ln(xy+x-x-1)$

(7) $f(x,y)=\sqrt{(x^2-4)(y^2-9)}$

(8) $f(x,y)=\dfrac{1}{\ln(4-x^2-y^2)}$

(9) $f(x,y)=\dfrac{\sqrt{y-x^2}}{1-x^2}$

(10) $f(x,y,z)=\sqrt{4-x^2}+\sqrt{4-y^2}+\sqrt{4-z^2}$

(11) $f(x,y)=\dfrac{1}{1+x^2y^2}$

Limits

Example [2] Find

(a) $\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2 y + 5xy - y^3}$ (b) $\lim_{(x,y)\to(3,-4)} \sqrt{x^2 + y^2}$

(c) $\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$ (d) $\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2 + y^2}$

(e) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$ (f) $\lim_{(x,y)\to(0,0)} \frac{2x^2 y}{x^4 + y^2}$

Solution

(a) $\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2 y + 5xy - y^3}$

$$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2 y + 5xy - y^3} = \frac{0 - 0\times 1 + 3}{0^2 \times 1 + 5\times 0\times 1 - 1^3} = -3$$

(b) $\lim_{(x,y)\to(3,-4)} \sqrt{x^2 + y^2}$

$$\lim_{(x,y)\to(3,-4)} \sqrt{x^2 + y^2} = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$$

(c) $\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \lim_{(x,y)\to(0,0)} \left(\frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \times \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}} \right)$$

$$= \lim_{(x,y)\to(0,0)} \left(\frac{x\cancel{(x - y)}\left(\sqrt{x} + \sqrt{y}\right)}{\cancel{x - y}} \right) = \lim_{(x,y)\to(0,0)} x\left(\sqrt{x} + \sqrt{y}\right) = 0(0+0) = 0$$

(d) $\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2 + y^2}$

Let $y = mx$

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2 + y^2}\Bigg|_{y=mx} = \lim_{(x,y)\to(0,0)} \frac{4x(mx)^2}{x^2 + (mx)^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2(4mx)}{x^2(1+m^2)} = \lim_{(x,y)\to(0,0)} \frac{4mx}{1+m^2} = \frac{4m(0)}{1+m^2} = \frac{0}{1+m^2} = 0$$

(e) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$

Let $y = mx$

$$\lim_{(x,y)\to(0,0)} \frac{4xy}{x^2 + y^2}\Bigg|_{y=mx} = \lim_{(x,y)\to(0,0)} \frac{4x(mx)}{x^2 + (mx)^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2(4m)}{x^2(1+m^2)} = \frac{4m}{1+m^2} \qquad Not - exist$$

This limit changes with each value of the slope m. There is therefore no single number we may call the limit of f as (x, y) approaches the origin. *The limit fails to exist*.

$(f)\quad \lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$

$Let \quad y = kx^2$

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}\Bigg|_{y=mx} = \lim_{(x,y)\to(0,0)} \frac{2x^2(kx^2)}{x^4+(kx^2)^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^4(2k)}{x^4(1+k^2)} = \frac{2k}{1+k^2} \qquad Not-exist$$

This limit changes with each value of the parameter ***k***. There is therefore no single number we may call the limit of f as (x, y) approaches the origin. *The limit fails to exist*.

➢ Continuity

➢ Definition *A function f(x, y) is continuous at the point if*

1- f is defined at (x_o, y_o)

2- $\lim_{(x,y)\to(x_o,y_o)} f(x,y)$ exists

3- $\lim_{(x,y)\to(x_o,y_o)} f(x,y) = f(x_o,y_o)$

A function is continuous if it is ***continuous*** at every point of its domain.

➢ *Exercises 3-2*

*H.W 1-*Find the limits :

(1) $\lim_{(x,y)\to(0,0)} \frac{3x^2-y^2+5}{x^2+y^2+2}$

(2) $\lim_{(x,y)\to(2,-3)} \left(\frac{1}{x}+\frac{1}{y}\right)^2$

(3) $\lim_{(x,y)\to(1,1)} \frac{x^2-2xy+y^2}{x-y}$

(4) $\lim_{(x,y)\to(1,1)} \frac{xy-y-2x+2}{x-y}$

(5) $\lim_{(x,y)\to(2,-4)} \frac{y+4}{x^2y-xy+4x^2-4x}$

(6) $\lim_{(x,y)\to(0,0)} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

(7) $\lim_{(x,y)\to(2,2)} \frac{x+y-4}{\sqrt{x+y}-2}$

(8) $\lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4}$

(9) $\lim_{(x,y)\to(4,3)} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1}$

(10) $\lim_{(x,y)\to(2,2)} \frac{x-y}{x^4-y^4}$

(11) $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

(12) $\lim_{(x,y)\to(1,-1)} \frac{x^3+y^3}{x+y}$

2- Partial Derivatives

Partial Derivatives of a Function of Two Variables

If f is a function of two variables, its partial derivatives are the functions f_x and f_y defined :

$$f_x(x,y)=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y(x,y)=\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

Notations for Partial Derivatives If $z=f(x,y)$, we write

$$f_x(x,y)=f_x=\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}f(x,y)=\frac{\partial z}{\partial x}=f_1=D_1f=D_xf$$

$$f_y(x,y)=f_y=\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}f(x,y)=\frac{\partial z}{\partial y}=f_2=D_2f=D_yf$$

Rule for Finding Partial Derivatives of $z=f(x,y)$

1. To find f_x, regard y as a constant and differentiate $f(x,y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x,y)$ with respect to y.

Functions of More than two Variables

In general, if u is a function of n variables, $u=f(x_1,x_2,x_3,x_4,x_5,....x_n)$, its partial derivative with respect to the ith variable x_i is :

$$f_x(x,y)=\lim_{h\to 0}\frac{f(x_1,....,x_{i-1},x_i+h,x_{i+1},....x_n)-f(x_1,x_2,x_3,x_4,x_5,....x_n)}{h}$$

$$\frac{\partial u}{\partial x_i}=\frac{\partial f}{\partial x_i}==f_{x_i}=D_if$$

Higher Derivatives

$$(f_x)_x=f_{xx}=f_{11}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}=\frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y=f_{xy}=f_{12}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y\,\partial x}=\frac{\partial^2 z}{\partial y\,\partial x}$$

$$(f_y)_x=f_{yx}=f_{21}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x\,\partial y}=\frac{\partial^2 z}{\partial x\,\partial y}$$

$$(f_y)_y=f_{yy}=f_{22}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}=\frac{\partial^2 z}{\partial y^2}$$

$$Note\quad f_{xy}=f_{yx}$$

$$(f_{xy})_y=f_{xyy}=f_{122}=\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y\,\partial x}\right)=\frac{\partial^3 f}{\partial y^2\partial x}=\frac{\partial^3 z}{\partial y^2\partial x}$$

Example [3] If $f(x,y)=x^3+x^2y^3-2y^2$, find $f_x(2,1)$ and $f_y(2,1)$

Solution

$$f_x(x,y)=\frac{\partial f}{\partial x}=3x^2+2xy^3$$

$$f_x(2,1)=\frac{\partial f}{\partial x}=3(2)^2+2\times 2\times 1^3=16$$

$$f_y(x,y)=\frac{\partial f}{\partial y}=3x^2y^2-4y$$

$$f_y(x,y)=\frac{\partial f}{\partial y}=3(2)^2(1)^2-4\times 1=8$$

Example [4] Calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

$a)-\quad f(x,y)=\frac{2y}{y+\cos x}$ $\qquad b)-\quad f(x,y)=\sin\left(\frac{x}{1+y}\right)$

Solution

$$a)-\quad f(x,y)=\frac{2y}{y+\cos x}$$

$$\frac{\partial f}{\partial x}=\frac{(y+\cos x)\frac{\partial}{\partial x}(2y)-2y\frac{\partial}{\partial x}(y+\cos x)}{(y+\cos x)^2}=\frac{(y+\cos x)(0)-2y(-\sin x)}{(y+\cos x)^2}=\frac{2y\sin x}{(y+\cos x)^2}$$

$$\frac{\partial f}{\partial y}=\frac{(y+\cos x)\frac{\partial}{\partial y}(2y)-2y\frac{\partial}{\partial y}(y+\cos x)}{(y+\cos x)^2}=\frac{(y+\cos x)(2)-2y(1)}{(y+\cos x)^2}=\frac{2\cos x}{(y+\cos x)^2}$$

$$b)-\quad f(x,y)=\sin\left(\frac{x}{1+y}\right)$$

$$\frac{\partial f}{\partial x}=\cos\left(\frac{x}{1+y}\right)\times\frac{\partial}{\partial x}\left[\frac{x}{1+y}\right]$$

$$\frac{\partial f}{\partial x}=\cos\left(\frac{x}{1+y}\right)\times\left[\frac{(1+y)\frac{\partial}{\partial x}(x)-x\frac{\partial}{\partial x}(1+y)}{(1+y)^2}\right]=\cos\left(\frac{x}{1+y}\right)\times\left[\frac{(1+y)(1)-x(0)}{(1+y)^2}\right]$$

$$=\cos\left(\frac{x}{1+y}\right)\times\left[\frac{\cancel{(1+y)}}{(1+y)^{\cancel{2}}}\right]=\left(\frac{1}{1+y}\right)\cos\left(\frac{x}{1+y}\right)$$

$$\frac{\partial f}{\partial y}=\cos\left(\frac{x}{1+y}\right)\times\left[\frac{(1+y)\frac{\partial}{\partial y}(x)-x\frac{\partial}{\partial y}(1+y)}{(1+y)^2}\right]=\cos\left(\frac{x}{1+y}\right)\times\left[\frac{(1+y)(0)-x(1)}{(1+y)^2}\right]$$

$$=\left(\frac{-x}{(1+y)^2}\right)\cos\left(\frac{x}{1+y}\right)$$

Example [5] Find $\frac{\partial z}{\partial x}$ if the equation

$a)-\quad yz-\ln z=x+y \qquad b)-\quad x^3+y^3+z^3+6xyz=1$

Solution

$a)-\quad yz-\ln z=x+y$

$$\frac{\partial}{\partial x}(yz)-\frac{\partial}{\partial x}(\ln z)=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial x}(y)$$

$$y\frac{\partial z}{\partial x}-\frac{1}{z}\frac{\partial z}{\partial x}=1+0$$

$$\frac{\partial z}{\partial x}\left(y-\frac{1}{z}\right)=1$$

$$\frac{\partial z}{\partial x}=\frac{z}{zy-1}$$

$b)-\quad x^3+y^3+z^3+6xyz=1$

$$\frac{\partial}{\partial x}(x^3)+\frac{\partial}{\partial x}(y^3)+\frac{\partial}{\partial x}(z^3)+\frac{\partial}{\partial x}(6xyz)=\frac{\partial}{\partial x}(1)$$

$$3x^2+0+3z^2\frac{\partial z}{\partial x}+6y\left(x\frac{\partial z}{\partial x}+z\right)=0$$

$$3x^2+3z^2\frac{\partial z}{\partial x}+6yx\frac{\partial z}{\partial x}+6yz=0$$

$$\left(3z^2+6yx\right)\frac{\partial z}{\partial x}=-\left(3x^2+6yz\right)$$

$$\frac{\partial z}{\partial x}=-\frac{x^2+2yz}{z^2+2yx}$$

Example [6] Find the second-order derivatives

$a)-\quad f(x,y)=x^3+x^2y^3-2y^2 \qquad b)-\quad f(x,y)=x\cos y+ye^x$

Solution

$a)-\quad f(x,y)=x^3+x^2y^3-2y^2$

$$f_x=\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(x^3+x^2y^3-2y^2\right)=3x^2+2xy^3$$

$$f_y=\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(x^3+x^2y^3-2y^2\right)=3x^2y^2-4y$$

$$f_{xx}=\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}(f_x)=\frac{\partial}{\partial x}\left(3x^2+2xy^3\right)=6x+2y^3$$

$$f_{xy}=\frac{\partial^2 f}{\partial x\,\partial y}=f_{yx}=\frac{\partial^2 f}{\partial y\,\partial x}=\frac{\partial}{\partial y}(f_x)=\frac{\partial}{\partial y}\left(3x^2+2xy^3\right)=6xy^2$$

$$f_{yy}=\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}(f_y)=\frac{\partial}{\partial y}\left(3x^2y^2-4y\right)=6x^2y-4$$

$b)-\quad f(x,y)=x\cos y+ye^{x}$

$$f_x=\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(x\cos y+ye^{x}\right)=\cos y+ye^{x}$$

$$f_y=\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(x\cos y+ye^{x}\right)=-x\sin y+e^{x}$$

$$f_{xx}=\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}(f_x)=\frac{\partial}{\partial x}\left(\cos y+ye^{x}\right)=ye^{x}$$

$$f_{xy}=\frac{\partial^2 f}{\partial x\,\partial y}=f_{yx}=\frac{\partial^2 f}{\partial y\,\partial x}=\frac{\partial}{\partial y}(f_x)=\frac{\partial}{\partial y}\left(\cos y+ye^{x}\right)=-\sin y+e^{x}$$

$$f_{yy}=\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}(f_y)=\frac{\partial}{\partial y}\left(-x\sin y+e^{x}\right)=-x\cos y$$

➤ *Exercises 3-3*

H.W 1- Calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$:

1) $f(x,y)=5xy-7x^2-y^2+3x-6y+2$

2) $f(x,y)=(xy-1)^2$

3) $f(x,y)=\sqrt{x^2+y^2}$

4) $f(x,y)=\frac{1}{x+y}$

5) $f(x,y)=\frac{x+y}{xy-1}$

6) $f(x,y)=e^{x+y+1}$

7) $f(x,y)=\ln(x+y)$

8) $f(x,y)=\sin^2(x-3y)$

9) $f(x,y)=\sin^{-1}(xy)$

10) $f(x,y)=e^{-x}\sin(x+y)$

*H.W 2-*Calculate the Second-Order Partial Derivatives

1) $f(x,y)=x+y+xy$

2) $f(x,y)=ye^{x^2}-y$

3) $f(x,y)=\sin xy$

4) $f(x,y)=\ln(x+y)$

5) $f(x,y)=x^2y+\cos y+y\sin x$

6) $f(x,y)=x\sin(x^2y)$

7) $f(x,y)=xe^{y}+y+1$

8) $f(x,y)=\sin^2(x-3y)$

9) $f(x,y)=\tan^{-1}\left(\frac{y}{x}\right)$

10) $f(x,y)=\frac{x-y}{x^2+y}$

3- the Chain Rule

If $y = f(x)$ and $x = g(t)$ where f and g are differentiable functions, then y is indirectly a differentiable function of t :

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

For functions of more than one variable, the Chain Rule has several versions, each of them giving a rule for differentiating a composite function.

➢ The Chain Rule (Case 1)

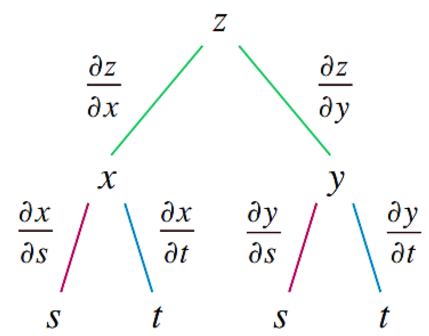
Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t :

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

➢ The Chain Rule (Case 2)

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are both differentiable functions of t. Then z is a differentiable function of t :

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$



➢ The Chain Rule (General Version)

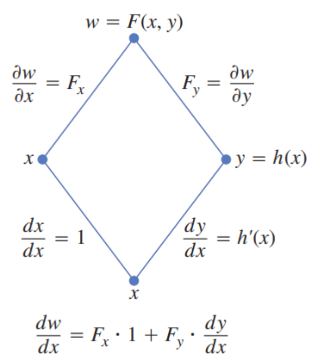
Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$:

$$\frac{\partial z}{\partial t_i} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \frac{\partial z}{\partial x_3}\frac{\partial x_3}{\partial t_i} + \ldots\ldots + \frac{\partial z}{\partial x_i}\frac{\partial x_i}{\partial t_i} + \ldots\ldots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

➢ A Formula for Implicit Differentiation

Suppose that $F(x, y)$ is differentiable and that the equation defines y as a differentiable function of x. Then at any point where $F_y \neq 0$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

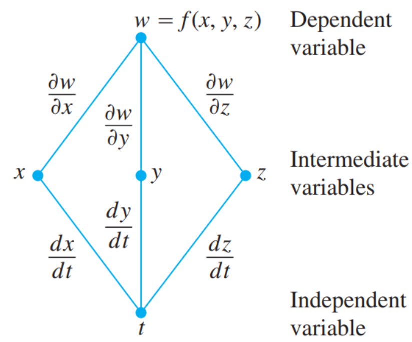


Example [7] Find (dw/dt) if $w = xy + z \quad x = \cos t \quad y = \sin t \quad z = t$

Solution

$w = xy + z \qquad x = \cos t \qquad y = \sin t \qquad z = t$

Chain Rule



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\frac{dw}{dt} = (y)(-\sin t) + (x)(\cos t) + (1)(1)$$

$$\frac{dw}{dt} = (\sin t)(-\sin t) + (\cos t)(\cos t) + 1$$

$$\frac{dw}{dt} = \sin^2 t + \cos^2 t + 1$$

$$\frac{dw}{dt} = 1 + \cos 2t$$

Example [8] Express $(\partial w / \partial r)$ and $(\partial w / \partial s)$ in terms of r and s if

$w = x + 2y + z^2 \qquad x = \dfrac{r}{s} \qquad y = r^2 + \ln s \qquad z = 2r$

Solution

$w = x + 2y + z^2 \qquad x = \dfrac{r}{s} \qquad y = r^2 + \ln s \qquad z = 2r$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial r} = (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2)$$

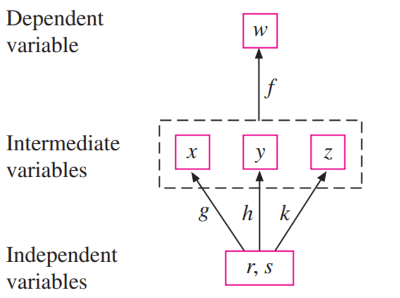
$$\frac{\partial w}{\partial r} = (1)\left(\frac{1}{s}\right) + (2)(2r) + (2 \times 2r)(2)$$

$$\frac{\partial w}{\partial r} = \frac{1}{s} + 12r$$

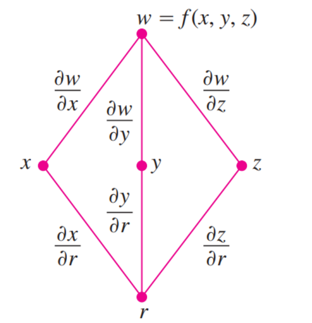
$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial w}{\partial s} = (1)\left(\frac{-r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0)$$

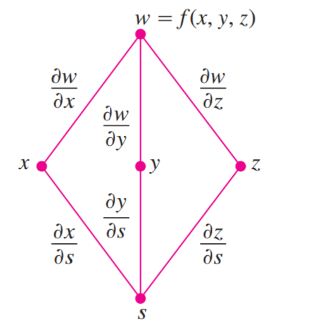
$$\frac{\partial w}{\partial s} = \frac{2}{s} - \frac{r}{s^2}$$



$w = f(g(r, s), h(r, s), k(r, s))$



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$



$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

Example [9] Find (dy/dx) if $y^2 - x^2 - \sin xy = 0$

Solution

$$F(x,y) = y^2 - x^2 - \sin xy$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = \frac{2x + y\cos xy}{2y - x\cos xy}$$

➢ *Exercises 3-4*

H.W 1- Evaluate $\frac{dw}{dt}$ at the given value of *t :*

1) $-\quad w = x^2 + y^2 \quad x = \cos t \quad y = \sin t$

2) $-\quad w = x^2 + y^2 \quad x = \cos t + \sin t \quad y = \cos t - \sin t$

3) $-\quad w = \frac{x}{z} + \frac{y}{z} \quad x = \cos^2 t \quad y = \sin^2 t \quad z = \frac{1}{t}$

4) $-\quad w = \ln\left(x^2 + y^2 + z^2\right) \quad x = \cos t \quad y = \sin t \quad z = 4\sqrt{t}$

5) $-\quad w = 2ye^x - \ln z \quad x = \ln\left(t^2 + 1\right) \quad y = \tan^{-1} t \quad z = e^t$

6) $-\quad w = z - \sin xy \quad x = t \quad y = \ln t \quad z = e^{t-1}$

Chapter Four

Directional Derivatives

Mr. Munther 2020-2021
Civil Engineering Department

1- Directional Derivatives

➢ Gradients

The gradient vector (gradient) of f(x, y) at a point is the vector

$$\nabla \mathbf{f} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\mathbf{i} + \frac{\partial \mathbf{f}}{\partial \mathbf{y}}\mathbf{j} + \frac{\partial \mathbf{f}}{\partial \mathbf{z}}\mathbf{k}$$

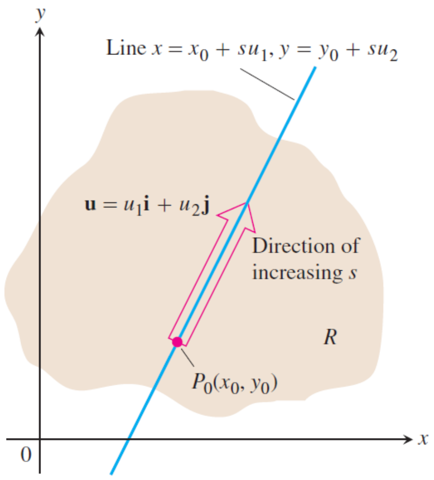
obtained by evaluating the partial derivatives of f at P_0.

➢ Directional Derivatives in the Plane

The derivative of f at $P_o(x_o, y_o, z_o)$ in the direction of the unit vector $\vec{\mathbf{A}} = \mathbf{a}_1\mathbf{i} + \mathbf{a}_2\mathbf{j} + \mathbf{a}_3\mathbf{k}$ is the directional derivative defined by $(D_u f)\big|_{P_o}$

If $f(x, y, z)$ is differentiable in an open region containing $P_o(x_o, y_o, z_o)$ then the dot product of the gradient $\nabla \mathbf{f}$ at $P_\circ$ and $\mathbf{u}$.

$$\left(\frac{\partial f}{\partial s}\right)_{u, P_o} = (D_u f)\big|_{P_o} = (\nabla f)\big|_{P_o} . \hat{\mathbf{u}}$$



Evaluating the dot product in the formula

$$D_u f = \nabla f \cdot u = |\nabla f| \cdot |u| \cos\theta = |\nabla f| \cos\theta$$

where θ is the angle between the vectors u and ∇f , reveals the following properties :

1. The function f increases when $(\theta = 0)$ $\quad \cos\theta = 1 \quad \Rightarrow \quad D_u f = |\nabla f|$
2. The function f decreases when $(\theta = \pi)$ $\quad \cos\theta = -1 \quad \Rightarrow \quad D_u f = -|\nabla f|$
3. The function f zero change when $(\theta = \pi / 2)$ $\quad \cos\theta = 0 \quad \Rightarrow \quad D_u f = 0$

Example [1] Find the derivative of $f(x,y,z)=cos(xy)+e^{yz}+ln(zx)$ at $P_o(1,0,1/2)$ in the direction of vector $\mathbf{A}=\mathbf{i}+2\mathbf{j}+2\mathbf{k}$.

Solution

The derivative of function $f(x,y,z)$ at point P_o in the direction of unit vector $\hat{u}=ai+bj+ck$ is the number :

$$(D_u f)\big|_{P_o}=(\nabla f)\big|_{P_o}.\hat{\mathbf{u}}$$

Since $(\nabla f)\big|_{P_o}=\left(\frac{\partial f}{\partial x}\right)\Big|_{P_o}\mathbf{i}+\left(\frac{\partial f}{\partial y}\right)\Big|_{P_o}\mathbf{j}+\left(\frac{\partial f}{\partial z}\right)\Big|_{P_o}\mathbf{k}$

Therefore, the ∇f is :

$$\left.\begin{aligned}\left(\frac{\partial f}{\partial x}\right)\Big|_{P_o}&=\left(-y\,sin(xy)+\frac{1}{x}\right)\Big|_{(1,0,\frac{1}{2})}=1\\ \left(\frac{\partial f}{\partial y}\right)\Big|_{P_o}&=\left(-x\,sin(xy)+ze^{yz}\right)\Big|_{(1,0,\frac{1}{2})}=\frac{1}{2}\\ \left(\frac{\partial f}{\partial z}\right)\Big|_{P_o}&=\left(ye^{yz}+\frac{1}{z}\right)\Big|_{(1,0,\frac{1}{2})}=2\end{aligned}\right\}\Rightarrow(\nabla f)\big|_{P_o}=\mathbf{i}+\frac{1}{2}\mathbf{j}+2\mathbf{k}$$

Since $\hat{u}=\frac{A}{|A|}=\frac{ai+bj+ck}{\sqrt{a^2+b^2+c^2}}=\frac{i+2j+2k}{\sqrt{1^2+2^2+2^2}}=\frac{1}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}$

$$(Duf)\big|_{P_o}=(\nabla f)\big|_{P_o}.\hat{u}=\left(\mathbf{i}+\frac{1}{2}\mathbf{j}+2\mathbf{k}\right).\left(\frac{1}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}\right)=(1\times\frac{1}{3})+(\frac{1}{2}\times\frac{2}{3})+(2\times\frac{2}{3})=2\Leftarrow\textbf{Ans.}$$

Example [2] Find the derivative of $f(x,y)=x\,tan^{-1}\left(\frac{y}{x}\right)$ at $P_o(1,1)$ in the direction of vector $\mathbf{A}=2\mathbf{i}-\mathbf{j}$.

Solution

$$\left.\begin{aligned}\left(\frac{\partial f}{\partial x}\right)\Big|_{P_o}&=\left(x\frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2}+tan^{-1}\left(\frac{y}{x}\right)\right)\Bigg|_{(1,1)}=\frac{\pi-2}{4}\\ \left(\frac{\partial f}{\partial y}\right)\Big|_{P_o}&=\left(x\frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}\right)\Bigg|_{(1,1)}=\frac{1}{2}\end{aligned}\right\}\Rightarrow(\nabla f)\big|_{P_o}=\frac{\pi-2}{4}\mathbf{i}+\frac{1}{2}\mathbf{j}$$

Since $\hat{u}=\frac{A}{|A|}=\frac{ai+bj}{\sqrt{a^2+b^2}}=\frac{2i-j}{\sqrt{2^2+1^2}}=\frac{2}{\sqrt{5}}\mathbf{i}-\frac{1}{\sqrt{5}}\mathbf{j}$

$$(Duf)\big|_{P_o}=(\nabla f)\big|_{P_o}.\hat{u}=\left(\frac{\pi-2}{4}\mathbf{i}+\frac{1}{2}\mathbf{j}\right).\left(\frac{2}{\sqrt{5}}\mathbf{i}-\frac{1}{\sqrt{5}}\mathbf{j}\right)=\frac{\pi-3}{2\sqrt{5}}\qquad\Leftarrow\textbf{Ans.}$$

Example [3] Find the derivative of $f(x,y)=\frac{x^2}{2}+\frac{y^2}{2}$ at $P_o(1,1)$, increases, decreases and zero change

Solution

$$(\nabla f)\big|_{P_o}=\left(\frac{\partial f}{\partial x}\right)\bigg|_{P_o}\mathbf{i}+\left(\frac{\partial f}{\partial y}\right)\bigg|_{P_o}\mathbf{j}$$

Therefore, the ∇f is :

$$\left.\begin{matrix}\left(\frac{\partial f}{\partial x}\right)\bigg|_{P_o}=(x)\big|_{(1,1)}=1\\ \left(\frac{\partial f}{\partial y}\right)\bigg|_{P_o}=(y)\big|_{(1,1)}=1\end{matrix}\right\}\Rightarrow(\nabla f)\big|_{P_o}=\mathbf{i}+\mathbf{j}$$

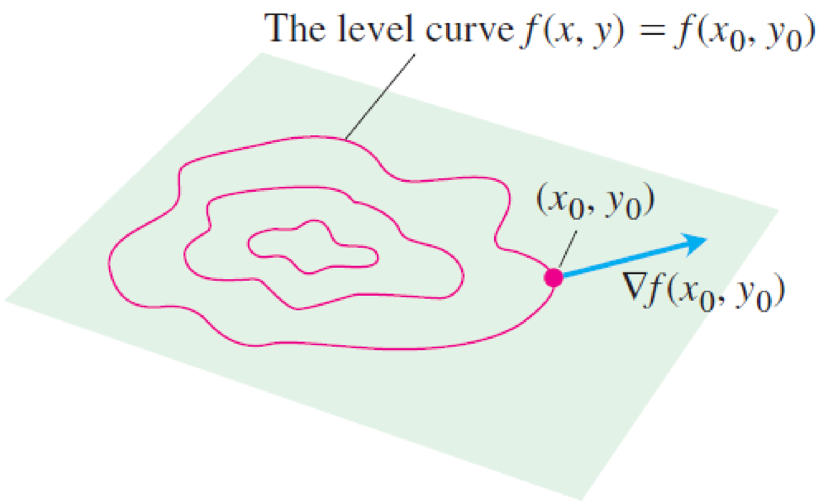
$$\hat{u}=\frac{\nabla f}{|\nabla f|}=\frac{i+j}{\sqrt{1^2+1^2}}=\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j}$$

1. The function f increases when $(\theta=0)$ $\quad cos\,\theta=1 \quad \Rightarrow \quad \hat{u}=\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j}$
2. The function f decreases when $(\theta=\pi)$ $\quad cos\,\theta=-1 \quad \Rightarrow \quad -\hat{u}=-\frac{1}{\sqrt{2}}\mathbf{i}-\frac{1}{\sqrt{2}}\mathbf{j}$
3. The function f zero change are the directions orthogonal when $(\theta=\pi/2)$

$$\bar{n}=-\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j} \qquad \& \qquad -\bar{n}=\frac{1}{\sqrt{2}}\mathbf{i}-\frac{1}{\sqrt{2}}\mathbf{j}$$

2- Tangents to Level Curves

At every point $P(x_o, y_o)$ in the domain of a differentiable function $f(x, y)$, the gradient of f is normal to the level curve through (x_o, y_o)



➢ Tangent Planes and Normal Lines

The tangent plane at the point $P_o(x_o, y_o, z_o)$ on the level surface $f(x, y, z) = c$ of a differentiable function f is the plane through P_o normal to $(\nabla f)|_{P_o}$

$$f_x(P_o)(x - x_o) + f_y(P_o)(y - y_o) + f_z(P_o)(z - z_o) = 0$$

the tangent line

$$f_x(P_o)(x - x_o) + f_y(P_o)(y - y_o) = 0$$

In surface $f(x, y) = z$

$$f_x(P_o)(x - x_o) + f_y(P_o)(y - y_o) + (z - z_o) = 0$$

The normal line of the surface at P_o is the line through P_o parallel to $(\nabla f)|_{P_o}$

$$x = x_o + f_x(P_o)t \qquad y = y_o + f_y(P_o)t \qquad z = z_o + f_z(P_o)t$$

Example [4] Find an equation for the tangent to the ellipse $\frac{x^2}{4} + y^2 = 2$ at the point (-2,1)

Solution

$$f_x(P_o) = \left(\frac{x}{2}\right)\Big|_{(-2,1)} = \frac{-2}{2} = -1$$

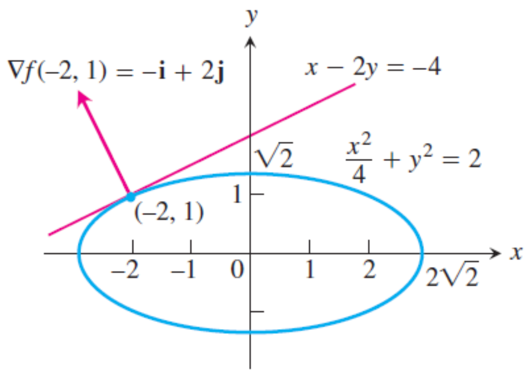
$$f_y(P_o) = (2y)|_{(-2,1)} = 2 \times 1 = 2$$

$$f_x(P_o)(x - x_o) + f_y(P_o)(y - y_o) = 0$$

$$(-1)(x - (-2)) + (2)(y - 1) = 0$$

$$-x - 2 + 2y - 2 = 0$$

$$x - 2y = -4$$



Example [5] Find the tangent plane and normal line of the surface at the point (1,2,4)
$f(x,y)=x^2+y^2+z-9$

Solution

x^2+y^2+z-9

$f_x(P_o)=(2x)|_{(1,2,4)}=2\times1=2$

$f_y(P_o)=(2y)|_{(1,2,4)}=2\times2=4$

$f_z(P_o)=(1)|_{(1,2,4)}=1$

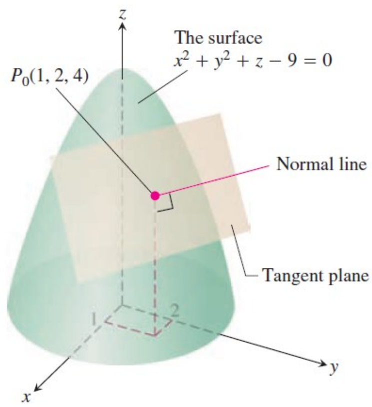
$f_x(P_o)(x-x_o)+f_y(P_o)(y-y_o)+f_z(P_o)(z-z_o)=0$

$(2)(x-1)+(4)(y-2)+(1)(z-4)=0$

$2x+4y+z=14$

$x=x_o+f_x(P_o)t \qquad y=y_o+f_y(P_o)t \qquad z=z_o+f_z(P_o)t$

$x=1+2t \qquad y=2+4t \qquad z=4+t$



Example [6] Find parametric equations for the line tangent to the curve of intersection of the surfaces at the point P(1,1,3) $x^2+y^2-2=0$ and $x+z-4=0$

Solution

$\vec{n}=\nabla f_1\times\nabla f_2$

$x^2+y^2-2=0$

$f_x(P_o)=(2x)|_{(1,1,3)}=2\times1=2$

$f_y(P_o)=(2y)|_{(1,1,3)}=2\times1=2$

$f_z(P_o)=(0)|_{(1,1,3)}=0$

$\nabla f_1=2i+2j$

$x+z-4=0$

$f_x(P_o)=(1)|_{(1,1,3)}=1$

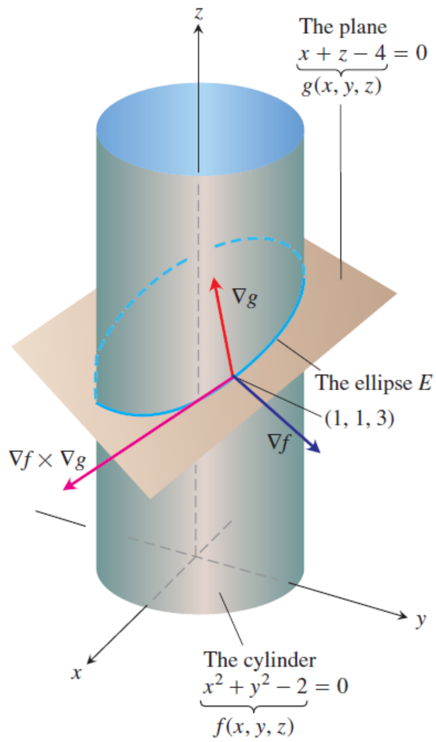
$f_y(P_o)=(0)|_{(1,1,3)}=0$

$f_z(P_o)=(1)|_{(1,1,3)}=1$

$$\vec{n}=\nabla f_1\times\nabla f_2=\begin{vmatrix} i & j & k \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix}=2i-2j-2k$$

$x=x_o+n_1t \qquad y=y_o+n_2t \qquad z=z_o+n_3t$

$x=1+2t \qquad y=1-2t \qquad z=3-2t$



Example [7] Find the derivative of $f(x,y)=x^2+y^2+z^2$ in the direction of the unit tangent vector of the helix $\mathbf{r}=(\cos t)\mathbf{i}+(\sin t)\mathbf{j}+t\,\mathbf{k}$ at $(t=-\pi/4)$

Solution

$$(Duf)\big|_{P_o}=(\nabla f)\big|_{P_o}.\hat{\mathbf{u}}$$

Since $(\nabla f)\big|_{P_o}=\left(\frac{\partial f}{\partial x}\right)\Big|_{P_o}\mathbf{i}+\left(\frac{\partial f}{\partial y}\right)\Big|_{P_o}\mathbf{j}+\left(\frac{\partial f}{\partial z}\right)\Big|_{P_o}\mathbf{k}$ Point $P_o\left(\cos\left(\frac{-\pi}{4}\right),\sin\left(\frac{-\pi}{4}\right),\frac{-\pi}{4}\right)=P_o\left(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}},\frac{-\pi}{4}\right)$

Therefore, the ∇f is :

$$\left.\begin{aligned}\left(\frac{\partial f}{\partial x}\right)\Big|_{P_o}&=(2x)\Big|_{\left(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}},\frac{-\pi}{4}\right)}=\sqrt{2}\\ \left(\frac{\partial f}{\partial y}\right)\Big|_{P_o}&=(2y)\Big|_{\left(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}},\frac{-\pi}{4}\right)}=-\sqrt{2}\\ \left(\frac{\partial f}{\partial z}\right)\Big|_{P_o}&=(2z)\Big|_{\left(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}},\frac{-\pi}{4}\right)}=\frac{-\pi}{2}\end{aligned}\right\}\Rightarrow(\nabla f)\big|_{P_o}=\sqrt{2}\,\mathbf{i}-\sqrt{2}\,\mathbf{j}-\frac{\pi}{2}\mathbf{k}$$

Since unit tangent vector r is :

$$\mathbf{T}=\frac{\mathrm{v}}{|\mathrm{v}|}=\frac{\frac{dr}{dt}}{\left|\frac{dr}{dt}\right|}=\frac{(-\sin t)\mathbf{i}+(\cos t)\mathbf{j}+t\,\mathbf{k}}{\sqrt{(-\sin t)^2+(\cos t)^2+1^2}}=\left(\frac{-\sin t}{\sqrt{2}}\right)\mathbf{i}+\left(\frac{\cos t}{\sqrt{2}}\right)\mathbf{j}+\frac{1}{\sqrt{2}}\mathbf{k}\quad,\quad\hat{\mathbf{u}}=\mathrm{T}\big|_{t=\frac{-\pi}{4}}=\frac{1}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}+\frac{1}{\sqrt{2}}\mathbf{k}$$

The derivative is :

$$(Duf)\big|_{P_o}=(\nabla f)\big|_{P_o}.\hat{\mathrm{u}}=\left(\sqrt{2}\,\mathbf{i}-\sqrt{2}\,\mathbf{j}-\frac{\pi}{2}\mathbf{k}\right).\left(\frac{1}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}+\frac{1}{\sqrt{2}}\mathbf{k}\right)=\frac{-\pi}{2\sqrt{2}}$$ $\Leftarrow$ **Ans.**

Example [8] The derivative of *f*(*x, y*) at a point $P_o(1,2)$ in the direction of the vector A = i + j is $2\sqrt{2}$ and the direction of the vector B = -2j is -3. What is the derivative of ***f*** in the direction of the vector C = -i -2j ?

Solution

The derivative of function $f(x,y)$ in the direction of the vector **v** is the number :

$$(Duf)\big|_{P_o}=(\nabla f)\big|_{P_o}.\hat{\mathbf{u}}$$

Since $\nabla f=f_x\mathbf{i}+f_y\mathbf{j}$ Therefore, $(Duf)\big|_{P_o}=(f_x\mathbf{i}+f_y\mathbf{j}).(u_1\mathbf{i}+u_2\mathbf{j})=f_xu_1+f_yu_2$

The $\hat{\mathbf{u}}$ is :

$$\hat{\mathrm{u}}_A=\frac{\mathrm{A}}{|\mathrm{A}|}=\frac{\mathrm{i}+\mathrm{j}}{\sqrt{1^2+1^2}}=\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j}$$

$$\hat{\mathrm{u}}_B=\frac{\mathrm{B}}{|\mathrm{B}|}=\frac{-2\mathrm{j}}{\sqrt{(-2)^2}}=-\mathbf{j}$$

$$\hat{\mathrm{u}}_C=\frac{\mathrm{C}}{|\mathrm{C}|}=\frac{-\mathrm{i}-2\mathrm{j}}{\sqrt{(-1)^2+(-2)^2}}=\frac{-1}{\sqrt{5}}\mathbf{i}-\frac{2}{\sqrt{5}}\mathbf{j}$$

Therefore,

$(Duf)|_{P_o} = f_x u_1 + f_y u_2$

$\text{at } \hat{\mathbf{u}}_A \Rightarrow Duf = 2\sqrt{2} = \frac{1}{\sqrt{2}} f_x + \frac{1}{\sqrt{2}} f_y \quad \Rightarrow \quad 4 = f_x + f_y \quad \ldots\ldots\ldots\ldots(1)$

$\text{at } \hat{\mathbf{u}}_B \Rightarrow Duf = -3 = 0 f_x + (-1) f_y \quad \Rightarrow \quad 3 = f_y \quad \ldots\ldots\ldots\ldots(2)$

$\therefore\ f_x = 1\ ,\ f_y = 3$

$\text{at } \hat{\mathbf{u}}_C \Rightarrow Duf = 1 \times \left(\frac{-1}{\sqrt{5}}\right) + 3 \times \left(\frac{-2}{\sqrt{5}}\right) = \frac{-7}{\sqrt{5}}$ $\Leftarrow$ **Ans.**

Example [9] In what two directions is the derivative of $f(x,y) = xy + y^2$ equal to zero at the point $(3,2)$?

Solution

Since the derivative of this function is equal to zero. Therefore, $(Duf)|_{P_o} = 0 \quad \Rightarrow \quad \hat{\mathbf{u}} \perp (\nabla f)|_{P_o}$

$$(\nabla f)|_{P_o} = \left(\frac{\partial f}{\partial x}\right)\bigg|_{P_o} \mathbf{i} + \left(\frac{\partial f}{\partial y}\right)\bigg|_{P_o} \mathbf{j} \ \Rightarrow\ (\nabla f)|_{P_o} = (y)|_{(3,2)}\mathbf{i} + (x+2y)|_{(3,2)}\mathbf{j} = 2\mathbf{i} + 7\mathbf{j}$$

$\mathbf{n} = -7\mathbf{i} + 3\mathbf{j}$ (Note: If $\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j}$, then $\mathbf{n}_1 = -b\,\mathbf{i} + a\,\mathbf{j}$ and $\mathbf{n}_2 = b\,\mathbf{i} - a\,\mathbf{j}$ are perpendicular to $\mathbf{v}$)

$\hat{\mathbf{u}}_1 = \frac{\mathbf{n}}{|\mathbf{n}|} = \frac{-7\mathrm{i} + 2\mathrm{j}}{\sqrt{(-7)^2 + 2^2}} = -\frac{7}{\sqrt{53}}\mathbf{i} + \frac{2}{\sqrt{53}}\mathbf{j}$ $\Leftarrow$ **Ans.**

$\hat{\mathbf{u}}_2 = -\hat{\mathbf{u}}_1 = \frac{7}{\sqrt{53}}\mathbf{i} - \frac{3}{\sqrt{53}}\mathbf{j}$ $\Leftarrow$ **Ans.**

Example [10] Shaw that the curve $\mathbf{r} = \sqrt{t}\,\mathbf{i} + \sqrt{t}\,\mathbf{j} + (2t-1)\mathbf{k}$ is tangent to the surface $x^2 + y^2 - z = 1$ when $t = 1$

Solution

In order to we are proved that the curve is tangent to the surface, if the result of the dot the derivative of the surface by vector of the curve equal to zero $\left(\nabla f|_{surface} \cdot \mathbf{v}|_{curve} = 0\right)$

$$\mathbf{v} = \frac{dr}{dt}\bigg|_{t_o} = \left(\frac{1}{2\sqrt{t}}\mathbf{i} + \frac{1}{2\sqrt{t}}\mathbf{j} + 2\mathbf{k}\right)\bigg|_{t=1} = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + 2\mathbf{k} \text{ Point } \left(\sqrt{t}, \sqrt{t}, +(2t-1)\right)\Big|_{t=1} \Rightarrow P_o(1,1,1)$$

$(\nabla f)|_{P_o} = (2x)|_{P_o}\mathbf{i} + (2y)|_{P_o}\mathbf{j} + (-1)|_{P_o}\mathbf{k} \quad \Rightarrow \nabla f = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

$\nabla f \cdot \mathbf{v} = (2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot \left(\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + 2\mathbf{k}\right) = \mathbf{0}$ $\Leftarrow$ **Ans.**

Example [11] Find the points on the surface $xy + yz + zx - x - z^2 = 0$ where the tangent plane is parallel to the *xy*-plane .

Solution

Since, (z = c) is equation of the plane parallel to the *xy*-plane Equation of the tangent plane

$$f_x(x - x_o) + f_y(y - y_o) + f_z(z - z_o) = 0$$

When the tangent plane is parallel to the *xy*-plane, therefore :

$$D_{\mathbf{i}} f = \nabla f \cdot \mathbf{i} = f_x = 0 \qquad \& \qquad D_{\mathbf{j}} f = \nabla f \cdot \mathbf{j} = f_y = 0$$

$$f_x = y + z - 1 = 0 \quad \Rightarrow \quad y = 1 - z \qquad \cdots\cdots(1)$$

$$f_y = x + z = 0 \quad \Rightarrow \quad x = -z \qquad \cdots\cdots(2)$$

Substituted in surface equation to we get the points :

$$(-z)(1-z) + (1-z)z + z(-z) - (-z) - z^2 = 0 \quad \Rightarrow \quad z - 2z^2 = 0$$

$$\therefore z = 0 \quad \& \quad z = \frac{1}{2}$$

$$P_1(0,1,0) \quad \& \quad p_2\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \qquad \Leftarrow \textbf{Ans.}$$

➢ Exercises 4-1

H.W 1- Find the gradient of the function at the given point.

(1) $f(x,y)=\ln(x^2+y^2)$ $(1,1)$

(2) $f(x,y)=\sqrt{2x+3y}$ $(-1,2)$

(3) $f(x,y)=\tan^{-1}\dfrac{\sqrt{x}}{y}$ $(4,-2)$

(4) $f(x,y,z)=x^2+y^2-2z^2+z\ln x$ $(1,1,1)$

(5) $f(x,y,z)=\sqrt{x^2+y^2+z^2}+\ln|xyx|$ $(-1,2,-2)$

(6) $f(x,y,z)=e^{x+y}\cos z+(y+1)\sin^{-1}x$ $(0,0,\pi/6)$

(7) $f(x,y,z)=xe^y+z^2$ $(1,\ln 2,1/2)$

(8) $f(x,y,z)=\ln|x^2+y^2-1|+y+6z$ $(1,1,0)$

H.W 2- Write the tangent line equation.

(1) $x^2+y^2=4$ $P_o(\sqrt{2},\sqrt{2})$

(2) $x^2-y=1$ $P_o(\sqrt{2},1)$

(3) $xy=-4$ $P_o(2,-2)$

(4) $x^2-xy+y^2=7$ $P_o(-1,2)$

(5) $xe^y-ye^x=5$ $P_o(4,1)$

H.W 3- Find the derivative of the function at P_o in the direction of u.

(1) $f(x,y)=2xy-3y^2$ $P_o(5,5)$ $u=4i+3j$

(2) $f(x,y)=\dfrac{x-y}{xy+2}$ $P_o(1,-1)$ $u=12i+5j$

(3) $f(x,y)=\tan^{-1}\left(\dfrac{y}{x}\right)+\sqrt{3}\sin^{-1}\left(\dfrac{xy}{2}\right)$ $P_o(1,1)$ $u=3i-2j$

(4) $f(x,y,z)=xy+yz+zx$ $P_o(1,-1,2)$ $u=3i+6j-2k$

(5) $f(x,y,z)=\cos xy+e^{yz}+\ln|zx|$ $P_o(1,0,1/2)$ $u=i+2j+2k$

H.W 4- Find equations for the tangent plane and normal line at the point on the surface

(1) $x^2+2xy-y^2+z^2=7$ $P_o(1,-1,3)$

(2) $\cos\pi x-x^2-xy-y^2-z=0$ $P_o(1,1,-1)$

H.W 5- Find parametric equations for the line tangent to the curve of intersection of the surfaces at the given point.

(1) $x^3+3x^2y^2+y^3+4xy-z^2=0$, $x^2+y^2+z^2=11$ $P_o(1,1,3)$

(2) $x^2+y^2=4$, $x^2+y^2-z=0$ $P_o(\sqrt{2},\sqrt{2},4)$

3- Estimating the Change in f in a Direction u

To estimate the change in the value of a differentiable function f when we move a small distance ds from a point P_o in a particular direction $\mathbf{u}$, use the formula

$$df = \underbrace{\left(\left(\nabla f\right)\Big|_{P_o} . \hat{\mathbf{u}} \right)}_{\text{Directional derivative}} \cdot \underbrace{ds}_{\text{Distance increment}}$$

Example [12] Estimate how much the value of $f(x,y,z) = y\sin x + 2yz$ will change if the point $P(x,y,z)$ moves 0.1 unit from $P_o(0,1,0)$ straight toward $P_1(2,2,-2)$

Solution

$\overrightarrow{P_oP_1} = 2i + j - 2k$ $\qquad u = \dfrac{\overrightarrow{P_oP_1}}{\left|\overrightarrow{P_oP_1}\right|} = \dfrac{2i + j - 2k}{\sqrt{2^2+1^2+2^2}} = \dfrac{2}{3}i + \dfrac{1}{3}j - \dfrac{2}{3}k$

$$f_x(P_o) = (y\cos x)\big|_{(0,1,0)} = 1\times 1 = 1$$

$$f_y(P_o) = (\sin x + 2z)\big|_{(0,1,0)} = 0 + 2\times 0 = 0$$

$$f_z(P_o) = (2y)\big|_{(0,1,0)} = 2\times 1 = 2$$

$$(\nabla f)\big|_{P_o} = i + 2k$$

$$(\nabla f)\big|_{P_o}\hat{u} = \frac{2}{3}\times 1 - \frac{2}{3}\times 2 = -\frac{2}{3}$$

$$df = \left((\nabla f)\big|_{P_o}.\hat{u}\right)\cdot ds = -\frac{2}{3}\times 0.1 = -0.06667\,unit$$

4- Linearization

The **linearization** of a function $f(x,y,z)$ at a point $P_o(x_o,y_o,z_o)$ where f is differentiable is the function

$$L(x,y,z) = f(P_o) + f_x(P_o)(x-x_o) + f_y(P_o)(y-y_o) + f_z(P_o)(z-z_o)$$

The approximation

$$f(x,y,z) \approx L(x,y,z)$$

is the standard linear approximation of f at (x_o,y_o,z_o).

Example [13] Find the linearization of $f(x,y) = x^2 - xy + \dfrac{y^2}{2} + 3$ at the point (3, 2).

Solution

$$f(P_o) = 3^2 - 3\times 2 + \frac{2^2}{2} + 3 = 8$$

$f_x(P_o) = (2x - y)\big|_{(3,2)} = 2\times 3 - 2 = 4$ $\qquad f_y(P_o) = (-x + y)\big|_{(3,2)} = -3 + 2 = -1$

$$L(x,y) = f(P_o) + f_x(P_o)(x-x_o) + f_y(P_o)(y-y_o)$$

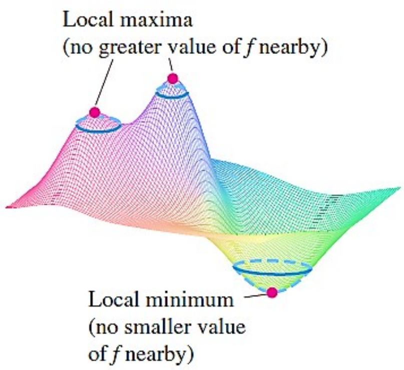
$L(x,y) = 8 + 4\times(x-3) + (-1)(y-2)$ $\qquad L(x,y) = 4x - y - 2$

5- Extreme Values and Saddle Points

Let *f*(x, y) be defined on a region R containing the point (a, b). Then

1. *f*(a, b) is a local maximum value of *f*(x, y) if for all domain points (x, y) in an open disk centered at (a, b).

2. *f*(a, b) is a local minimum value of *f*(x, y) if for all domain points (x, y) in an open disk centered at (a, b).



➢ First Derivative Test for Local Extreme Values

If *f*(x, y) has a local maximum or minimum value at an interior point (a, b) of its domain and if the first partial derivatives exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

➢ Second Derivative Test for Local Extreme Values

Suppose that *f*(x, y) and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Then $H = (f_{xx} f_{yy} - f_{xy}^2)$

1. *f*(x, y) has a ***local maximum*** at (a, b) if $f_{xx} < 0$ and $H > 0$ at (a, b).
2. *f*(x, y) has a ***local minimum*** at (a, b) if $f_{xx} > 0$ and $H > 0$ at (a, b).
3. *f*(x, y) has a ***saddle point*** at (a, b) if $H < 0$ at (a, b).
4. The test is ***inconclusive*** at (a, b) if $H = 0$ at (a, b). In this case, we must find some other way to determine the behavior of f at (a, b).

Example [14] Find the local extreme values of the function

$$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4$$

Solution

$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4$

$f_x = y - 2x - 2 = 0$ ......(1)

$f_y = x - 2y - 2 = 0$ ......(2)

Solve Critical Point $(x, y) = (-2, -2)$

$f_{xx} = -2$ $f_{xy} = 1$ $f_{yy} = -2$

$H = f_{xx} f_{yy} - f_{xy}^2 = (-2)(-2) - (1)^2 = 3$

$\because H > 0$ & $f_{xx} < 0$

$\therefore$ The f has a local maximum at $(-2, -2)$.

Example [15] Find the local extreme values of the function

$$f(x,y)=3y^2-2y^3-3x^2+6xy$$

Solution

$f(x,y)=3y^2-2y^3-3x^2+6xy$

$f_x=-6x+6y=0 \quad \Rightarrow \quad x=y$

$f_y=6y-6y^2+6x=0 \quad \Rightarrow \quad 6y-6y^2+6y=0$

$6y(2-y)=0 \quad y=0 \;\; or \;\; y=2$

The two critical points $(0,0)$ & $(2,2)$

$f_{xx}=-6 \qquad f_{xy}=6 \qquad f_{yy}=6-12y$

$H=f_{xx}f_{yy}-f_{xy}^2=(-6)(6-12y)-(6)^2=72(y-1)$

Test a point $(0,0)$

$H=72(y-1)=72(0-1)=-72$

$\because H<0$

$\therefore$ The f has a saddle point at $(0,0)$.

Test a point $(2,2)$

$H=72(y-1)=72(2-1)=72$

$\because H>0 \quad \& \quad f_{xx}<0$

$\therefore$ The f has a local maximum at $(2,2)$.

6- Lagrange Multipliers

The method of Lagrange multipliers a powerful method for finding extreme values of constrained functions.

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq 0$ when $g(x, y, z)=0$. To find the local maximum and minimum values of f subject to the constraint $g(x, y, z)$ (if these exist), find the values of x, y, z, and λ that simultaneously satisfy the equations

$$\nabla f=\lambda \nabla g \qquad \text{and} \qquad g(x,y,z)=0$$

For functions of two independent variables, the condition is similar, but without the variable z.

Example [16] Find the greatest and smallest values that the function $f(x,y)=xy$ takes on the ellipse $\frac{x^2}{8}+\frac{y^2}{2}=1$

Solution

$$g(x,y)=\frac{x^2}{8}+\frac{y^2}{2}-1=0$$

$$\nabla f=f_x i+f_y j=yi+xj \qquad \nabla g=g_x i+g_y j=\left(\frac{x}{4}\right)i+(y)j \qquad \nabla f=\lambda\nabla g$$

$$yi+xj=\lambda\left(\left(\frac{x}{4}\right)i+(y)j\right) \qquad yi+xj=\left(\frac{\lambda x}{4}\right)i+(\lambda y)j$$

$$y=\frac{\lambda x}{4} \qquad x=\lambda y \qquad \Rightarrow \qquad y=\frac{\lambda(\lambda y)}{4}=\frac{\lambda^2}{4}y$$

$$\frac{\lambda^2}{4}y-y=0 \qquad y\left(\frac{\lambda^2}{4}-1\right)=0$$

$$y=0 \qquad \Rightarrow \qquad (0,0)$$

$$or \qquad \lambda=\pm2 \qquad \Rightarrow \qquad x=\pm2y$$

$$\frac{(\pm2y)^2}{8}+\frac{y^2}{2}=1 \qquad \Rightarrow \qquad y=\pm1 \qquad \Rightarrow \qquad (\pm2,1),(\pm2,-1)$$

The origin point (0, 0) is not on the ellipse. The function therefore takes on its extreme values on the ellipse at the four points $(\pm2,1),(\pm2,-1)$. The extreme values are $xy=2$ and $xy=-2$.

Example [17] Find the maximum and minimum values of the function $f(x,y)=3x+4y$ on the circle $x^2+y^2=1$

Solution

$$g(x,y)=x^2+y^2-1=0$$

$$\nabla f=f_x i+f_y j=3i+4j$$

$$\nabla g=g_x i+g_y j=(2x)i+(2y)j$$

$$\nabla f=\lambda\nabla g \qquad 3i+4j=\lambda((2x)i+(2y)j)$$

$$3=2\lambda x \qquad x=\frac{3}{2\lambda} \qquad 4=2\lambda y \qquad y=\frac{2}{\lambda}$$

$$\left(\frac{3}{2\lambda}\right)^2+\left(\frac{2}{\lambda}\right)^2=1 \qquad \Rightarrow \qquad \lambda=\pm\frac{5}{2}$$

$$x=\frac{3}{2\lambda}=\pm\frac{3}{5} \qquad y=\frac{2}{\lambda}=\pm\frac{4}{5} \qquad \pm\left(\frac{3}{5},\frac{4}{5}\right)$$

We see that its maximum and minimum values on the circle $x^2+y^2=1$ are :

$$f(x,y)=3\times\frac{3}{5}+4\times\frac{4}{5}=5 \qquad f(x,y)=-3\times\frac{3}{5}-4\times\frac{4}{5}=-5$$

➢ *Exercises 4-2*

H.W 1- Estimating Change. By about how much will function $f(x,y)=\ln\left|\sqrt{x^2+y^2+z^2}\right|$ change if the point P(x, y, z) moves from $P_o(3, 4, 12)$ a distance of ds =0.1 unit in the direction of 3i + 6j - 2k.

H.W 2- Find the linearization of the function at each point.

(1) $f(x,y)=x^2+y^2+1$ $\quad P(1,1)$

(2) $f(x,y)=x^3y^4$ $\quad P(1,1)$

(3) $f(x,y)=e^x\cos y$ $\quad P\left(0,\frac{\pi}{2}\right)$

(4) $f(x,y)=e^{2y-x}$ $\quad P(1,2)$

(5) $f(x,y,z)=x^2+y^2+z^2$ $\quad P(0,1,0)$

(6) $f(x,y,z)=\frac{\sin(xy)}{z}$ $\quad P(2,0,1)$

(7) $f(x,y,z)=e^x+\cos(y+z)$ $\quad P\left(0,\frac{\pi}{4},\frac{\pi}{4}\right)$

(8) $f(x,y,z)=\tan^{-1}(xyz)$ $\quad P(1,1,0)$

H.W 3- Find all the local maxima, local minima, and saddle points of the functions

(9) $f(x,y)=x^2+xy+y^2+3x-3y+4$

(10) $f(x,y)=x^3+3xy+y^3$

(11) $f(x,y)=4xy-x^4-y^4$

(12) $f(x,y)=\frac{1}{x^2+y^2-1}$

(13) $f(x,y)=e^{x^2+y^2-4x}$

(14) $f(x,y)=e^{-y}(x^2+y^2)$

(15) $f(x,y)=x^4y^4$

(16) $f(x,y)=\ln|x+y|+x^2-y$

H.W 4- Extrema on an ellipse Find the points on the ellipse $x^2+2y^2=1$ where $f(x,y)=xy$ has its extreme values.

H.W 5- Maximum on a line Find the maximum value of $f(x,y)=49-x^2-y^2$ on the line $x+3y=10$

H.W 6- Extrema on a line Find the local extreme values of $f(x,y)=x^2y$ on the line $x+y=3$

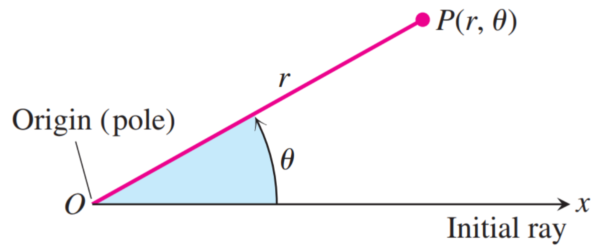
Chapter Five

Polar Coordinate

Mr. Munther 2020-2021
Civil Engineering Department

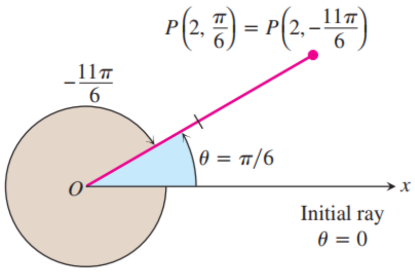
1- Polar Coordinate

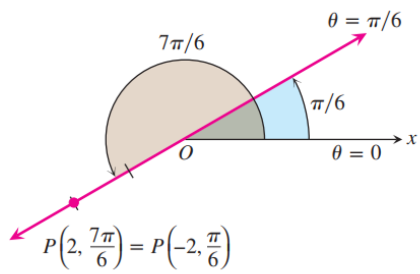
To define ***polar coordinates***, we first fix an ***origin O*** (called the pole) and an ***initial ray*** from ***O***. Then each point ***P*** can be located by assigning to it a polar coordinate pair (r,θ) in which ***r*** gives the directed distance from ***O*** to ***P*** and θ gives the directed angle from the initial ray to ray ***OP***.



$P(r,\theta)$

Directed distance from O to P ← → *Directed angle from initial ray to OP*





Example [1] Find all the polar coordinates of the point $P(2,\pi/6)$

Solution

For r=2, the angles are

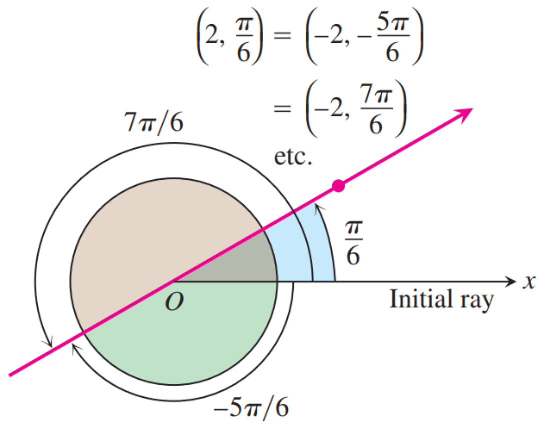
$$\frac{\pi}{6},\frac{\pi}{6}\pm 2\pi,\frac{\pi}{6}\pm 4\pi,\frac{\pi}{6}\pm 6\pi......$$

$$\left(2,\frac{\pi}{6}+2n\pi\right) \quad n=0,\pm1,\pm2,\pm3,......$$

For r=-2, the angles are

$$-\frac{5\pi}{6},-\frac{5\pi}{6}\pm 2\pi,-\frac{5\pi}{6}\pm 4\pi,-\frac{5\pi}{6}\pm 6\pi......$$

$$\left(2,-\frac{5\pi}{6}+2n\pi\right) \quad n=0,\pm1,\pm2,\pm3,......$$



2- Equations Relating Polar and Cartesian Coordinates

➢ Definition

x-coordinate	$x = r\cos\theta$
y-coordinate	$y = r\sin\theta$
r-coordinate	$r^2 = x^2 + y^2$
θ-angle	$\tan\theta = \frac{y}{x}$

Example [2] Find a polar equation for the circle $x^2+(y-3)^2=9$

Solution

$x^2+(y-3)^2=9$

$x^2+y^2-6y+9=9$

$r^2-6r\sin\theta=0$

$r=6\sin\theta$

Example [3] Replace the polar equation by equivalent Cartesian equations $r=4\cos\theta$

Solution

$r=4\cos\theta \qquad \times r$

$r^2=4r\cos\theta$

$x^2+y^2=4x$

$x^2-4x+4-4+y^2=0$

$(x-2)^2+y^2=4$

➢ *Exercises 5-1*

*H.W 1-*Replace the polar equations with equivalent Cartesian equations.

1]. $r=4\sin\theta$

2]. $r^2\sin 2\theta=2$

3]. $r=4\tan\theta\sec\theta$

4]. $r=\cot\theta\csc\theta$

5]. $r=\cot\theta\csc\theta$

6]. $r\sin\theta=\ln r+\ln\cos\theta$

7]. $r=\csc\theta e^{r\cos\theta}$

8]. $r=\dfrac{5}{\sin\theta-2\cos\theta}$

*H.W 2-*Replace the Cartesian equations with equivalent polar equations.

1]. $x^2-y^2=1$

2]. $xy=2$

3]. $(x+2)^2+(y-5)^2=16$

4]. $y^2=4x$

5]. $x-y=3$

6]. $\sqrt{x^2+y^2}=xy$

7]. $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$

8]. $\dfrac{y^2}{9}-\dfrac{x^2}{4}=1$

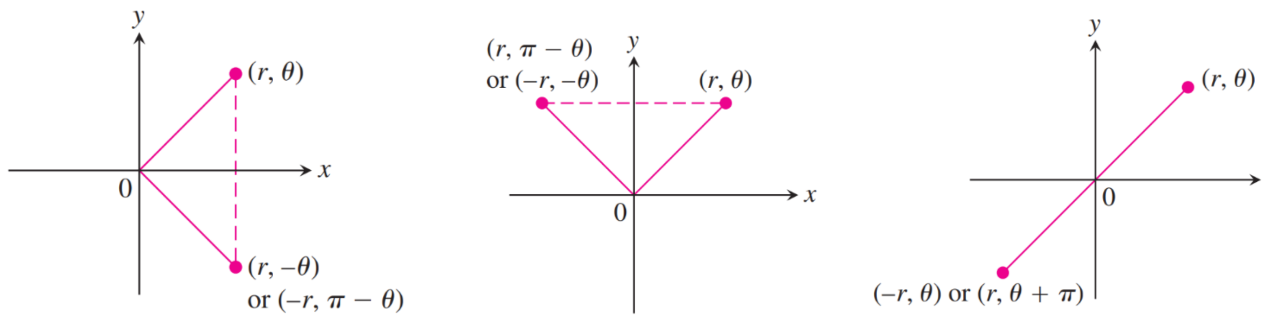
3- Graphing in Polar Coordinates

Symmetry

1. *Symmetry about the x-axis:* If the point (r,θ) lies on the graph, then the point $(r,-\theta)$ or $(-r,\pi-\theta)$ lies on the graph .

2. *Symmetry about the y-axis:* If the point (r,θ) lies on the graph, then the point $(r,\pi-\theta)$ or $(-r,-\theta)$ lies on the graph .

3. *Symmetry about the origin:* If the point (r,θ) lies on the graph, then the point $(-r,\theta)$ or $(r,\theta+\pi)$ lies on the graph .



Slope of the Curve

$$\left.\frac{dy}{dx}\right|_{(r,\theta)}=\frac{f'(\theta)\sin\theta+f(\theta)\cos\theta}{f'(\theta)\cos\theta-f(\theta)\sin\theta}$$

Example [4] Graph the curve $r=1-\cos\theta$

Solution

Symmetry

$x-axis \quad (r,\theta)=(r,-\theta)$

$r=1-\cos(-\theta) \quad \Rightarrow \quad r=1-\cos\theta \quad O.K$

$y-axis \quad (r,\theta)=(-r,-\theta)$

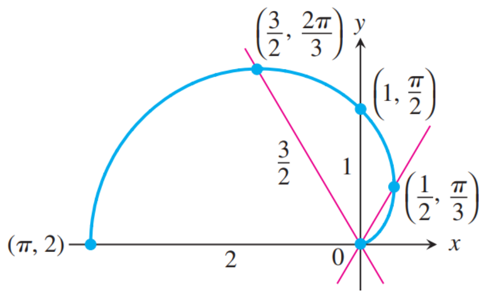
$-r=1-\cos(-\theta) \quad \Rightarrow \quad r=-1+\cos\theta \quad Not\ O.K$

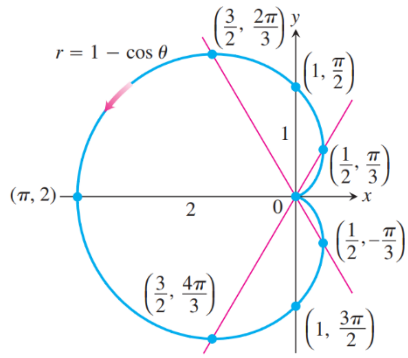
$Origin \quad (r,\theta)=(-r,\theta)$

$-r=1-\cos\theta \quad \Rightarrow \quad r=-1+\cos\theta \quad Not\ O.K$

θ	r
0	0
π/3	1/2
π/2	1
2π/3	3/2
π	2

The curve is Symmetry about the x-axis





Example [5] Graph the curve $r^2 = 4\cos\theta$

Solution

Symmetry

$x-axis \quad (r,\theta)=(r,-\theta)$

$r^2 = 4\cos(-\theta) \quad \Rightarrow \quad r^2 = 4\cos\theta \quad O.K$

$y-axis \quad (r,\theta)=(-r,-\theta)$

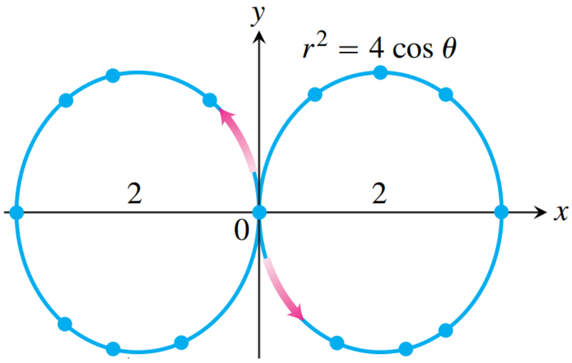
$(-r)^2 = 4\cos(-\theta) \quad \Rightarrow \quad r^2 = 4\cos\theta \quad O.K$

$Origin \quad (r,\theta)=(-r,\theta)$

$(-r)^2 = 4\cos\theta \quad \Rightarrow \quad r^2 = 4\cos\theta \quad O.K$

θ	r
0	2
π/6	1.9
π/4	1.7
π/3	1.4
π/2	0

The curve is Symmetry about the x-axis, y-axis and origin



Example [6] Find the intersections of the curves $r = \sin\theta$ and $r = 1 - \sin\theta$

Solution

$r = \sin\theta \qquad r = 1 - \sin\theta$

$r_1 = r_2$

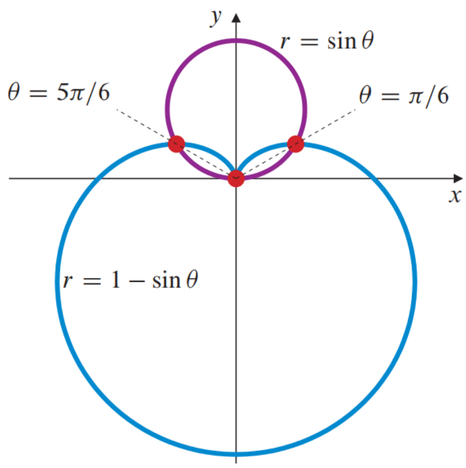
$\sin\theta = 1 - \sin\theta$

$2\sin\theta = 1$

$\sin\theta = \frac{1}{2} \qquad \theta = \sin^{-1}\frac{1}{2}$

$\therefore \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\therefore r = \frac{1}{2}$



Exercises 5-2

H.W 1- Replace the polar equations with equivalent Cartesian equations.

1]. $r = 2 + \sin\theta$

2]. $r = \sin\frac{\theta}{2}$

3]. $r^2 = \cos\theta$

4]. $r^2 = -\sin\theta$

5]. $r = 2 - 2\cos\theta$

6]. $r = 1 + 2\sin\theta$

7]. $r = \cos\frac{\theta}{2}$

8]. $r^2 = \sin\theta$

9]. $r^2 = 4\sin 2\theta$

10]. $r^2 = -\cos 2\theta$

11]. $r = \frac{1}{\theta}$

12]. $r^2\theta = 1$

13]. $r = \theta^2$

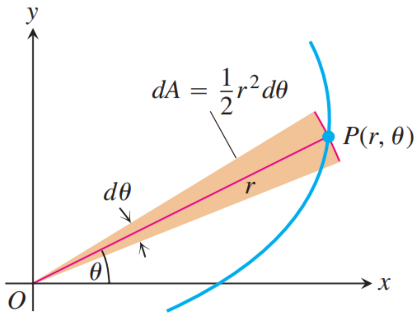
14]. $r = \ln\theta$

4- Areas in Polar Coordinates

Area of the Shaped Region Between the Origin and the Curve

$r = f(\theta) \quad , \quad \alpha \le \theta \le \beta$

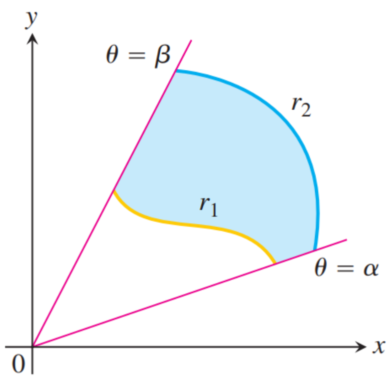
$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$



Area of the Region

$0 \le r_1(\theta) \le r \le r_2(\theta) \quad , \quad \alpha \le \theta \le \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta$$



Example [7] Find the area of the region in the plane enclosed by $r=2(1+\cos\theta)$

Solution

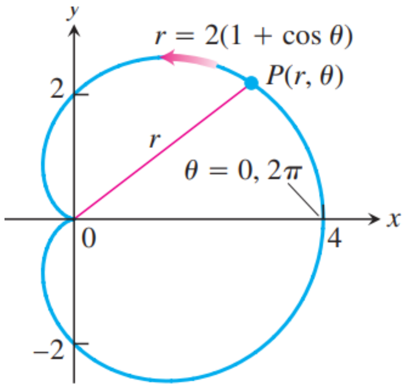
$r=2(1+\cos\theta)$

$$A=\int_{\alpha}^{\beta}\frac{1}{2}r^2d\theta=\int_{0}^{2\pi}\frac{1}{2}[2(1+\cos\theta)]^2\,d\theta$$

$$=2\int_{0}^{2\pi}(1+\cos\theta)^2\,d\theta$$

$$=2\int_{0}^{2\pi}(1+2\cos\theta+\cos^2\theta)d\theta=2\int_{0}^{2\pi}\left(1+2\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right)d\theta$$

$$=\int_{0}^{2\pi}(3+4\cos\theta+\cos 2\theta)d\theta=\left(3\theta+4\sin\theta+\frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi}=6\pi \quad unit^2$$

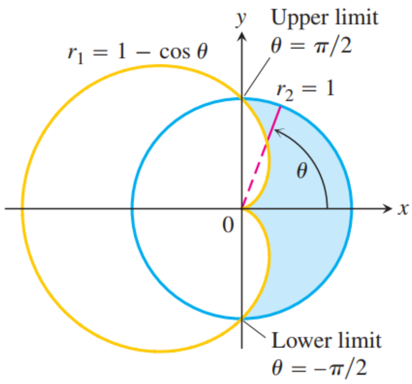


Example [8] Find the area inside the circle r=1 and outside the $r=1-\cos\theta$

Solution

$r_1=r_2$

$1=1-\cos\theta \qquad \cos\theta=0 \qquad \theta=\frac{\pi}{2},-\frac{\pi}{2}$



$$A=\int_{\alpha}^{\beta}\frac{1}{2}(r_2^2-r_1^2)d\theta=A=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{2}\left(1^2-(1-\cos\theta)^2\right)d\theta$$

$$=2\int_{0}^{\frac{\pi}{2}}\frac{1}{2}(1-1+2\cos\theta-\cos^2\theta)d\theta=\int_{0}^{\frac{\pi}{2}}(2\cos\theta-\cos^2\theta)d\theta$$

$$=\int_{0}^{\frac{\pi}{2}}\left(2\cos\theta-\frac{1}{2}[1+\cos 2\theta]\right)d\theta=\left(2\sin\theta-\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right)\Big|_0^{\frac{\pi}{2}}=2-\frac{\pi}{4}$$

5- Length of a Polar Curve

$$r=f(\theta) \qquad , \qquad \alpha\le\theta\le\beta$$

$$L=\int_{\alpha}^{\beta}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

Example [9] Find the total length of the cardioid $r = a(1-\cos\theta)$

Solution

$r = a(1-\cos\theta)$ $\qquad \dfrac{dr}{d\theta} = a\sin\theta$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{(a(1-\cos\theta))^2 + (a\sin\theta)^2}\, d\theta$$

$$= a\int_0^{2\pi} \sqrt{1-2\cos\theta+\cos^2\theta+\sin^2\theta}\; d\theta$$

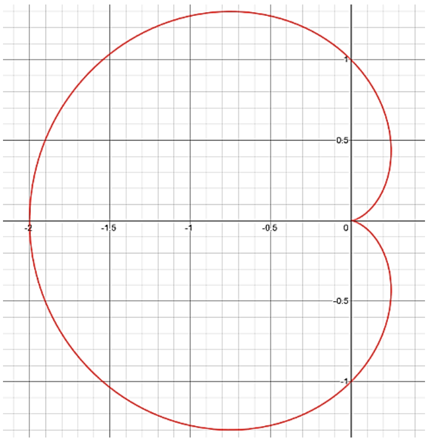
$$= a\int_0^{2\pi} \sqrt{2-2\cos\theta}\; d\theta =$$

$$= 2a\int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\; d\theta$$

$$= 2a\int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta = -4a\left(\cos\frac{\theta}{2}\right)\Big|_0^{2\pi}$$

$$= -8a(-1-1)$$

$$= 8a$$



➢ *Exercises 5-3*

H.W 1- Find the areas of the regions

1. Shared by the circles $r = 2\cos\theta$ and $r = 2\sin\theta$
2. Shared by the circles $r = 1$ and $r = 2\sin\theta$
3. Shared by the circle $r = 2$ and the cardioid $r = 2(1-\cos\theta)$
4. Shared by the cardioids $r = 2(1+\cos\theta)$ and $r = 2(1-\cos\theta)$
5. Inside the lemniscate $r^2 = 6\cos 2\theta$ and outside the circle $r = \sqrt{3}$

H.W 2- Find the lengths of the curves in

1. $r = \theta^2 \qquad 0 \le \theta \le \sqrt{5}$
2. $r = 1+\cos\theta$
3. $r = \dfrac{6}{1+\cos\theta} \qquad 0 \le \theta \le \dfrac{\pi}{2}$
4. $r = \cos^3\dfrac{\theta}{3} \qquad 0 \le \theta \le \dfrac{\pi}{4}$
5. $r = \sqrt{1+\sin 2\theta} \qquad 0 \le \theta \le \pi\sqrt{2}$

Chapter Six

Multiple Integral

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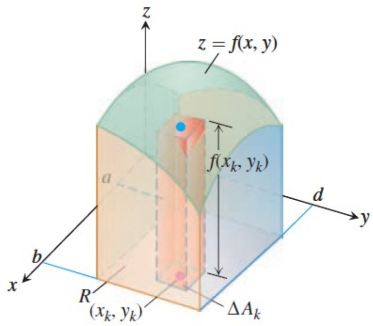
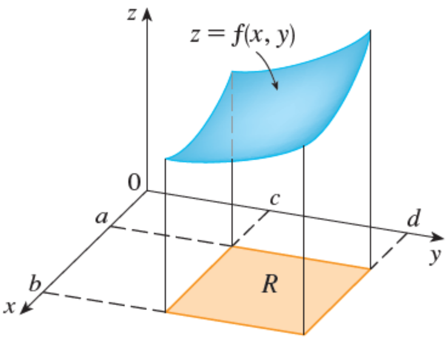
1. Double and Iterated Integrals over Rectangles

We consider a function ƒ(x, y) defined on a rectangular region R. When ƒ(x, y) is a positive function over a rectangular region R in the xy-plane, we may interpret the double integral of ƒ over R as the *volume* of the 3-dimensional solid region over the xy-plane bounded below by R and above by the surface.





$$Volume = \iint_R f(x,y)\,dA$$

$$= \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

$$= \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Example [1] Evaluate the iterated (repeated) and the double integrals

$a)\ \int_0^3 \int_1^2 x^2 y\,dy\,dx \qquad b)\ \int_{-3}^{3} \int_0^{\pi/2} \left(y + y^2 \cos x\right) dx\,dy \qquad c)\ \int_1^2 \int_1^5 \frac{\ln y}{xy}\,dy\,dx$

Solution

$a)\ \int_0^3 \int_1^2 x^2 y\,dy\,dx$

$\int_0^3 \int_1^2 x^2 y\,dy\,dx$

$$= \int_0^3 \left[x^2 \left(\frac{y^2}{2} \right) \Bigg|_1^2 \right] dx = \int_0^3 \left[x^2 \left(\frac{2^2}{2} - \frac{1^2}{2} \right) \right] dx = \frac{3}{2} \int_0^3 x^2 dx = \frac{3}{2} \times \left(\frac{x^3}{3} \right) \Bigg|_0^3 = \frac{3}{2} \times \left(\frac{3^3}{3} - \frac{0^3}{3} \right) = \frac{27}{2}$$

$\int_1^2 \int_0^3 x^2 y\,dx\,dy$

$$= \int_1^2 \left[y \left(\frac{x^3}{3} \right) \Bigg|_0^3 \right] dy = \int_1^2 \left[y \left(\frac{3^3}{3} - \frac{0^3}{3} \right) \right] dy = 9 \int_1^2 y\,dy = 9 \times \left(\frac{y^2}{2} \right) \Bigg|_1^2 = 9 \times \left(\frac{2^2}{2} - \frac{1^2}{2} \right) = \frac{27}{2}$$

$$b)\ \int_{-3}^{3}\int_{0}^{\pi/2}\left(y+y^2\cos x\right)dx\,dy$$

$$\int_{-3}^{3}\int_{0}^{\pi/2}\left(y+y^2\cos x\right)dx\,dy$$

$$=\int_{-3}^{3}\left[\left(yx+y^2\sin x\right)\Big|_0^{\pi/2}\right]dy=\int_{-3}^{3}\left[\left(y\times\frac{\pi}{2}+y^2\sin\frac{\pi}{2}\right)\right]dy=\int_{-3}^{3}\left[\left(\frac{\pi}{2}y+y^2\right)\right]dy$$

$$=\left(\frac{\pi y^2}{4}+\frac{y^3}{3}\right)\Big|_{-3}^{3}=\left(\frac{\pi 3^2}{4}+\frac{3^3}{3}\right)-\left(\frac{\pi(-3)^2}{4}+\frac{(-3)^3}{3}\right)=18$$

$$\int_{0}^{\pi/2}\int_{-3}^{3}\left(y+y^2\cos x\right)dy\,dx=\int_{0}^{\pi/2}\left[\left(\frac{y^2}{2}+\frac{y^3}{3}\cos x\right)\Big|_{-3}^{3}\right]dx=\int_{0}^{\pi/2}\left[\left(\frac{3^2}{2}+\frac{3^3}{3}\cos x\right)-\left(\frac{(-3)^2}{2}+\frac{(-3)^3}{3}\cos x\right)\right]dx$$

$$=\int_{0}^{\pi/2}18\cos x\,dx=18\left(\sin x\right)\Big|_0^{\pi/2}=18$$

$$c)\ \int_{1}^{2}\int_{1}^{5}\frac{\ln y}{xy}dy\,dx$$

$$=\int_{1}^{2}\frac{1}{2x}\left(\left(\ln y\right)^2\right)\Big|_1^5 dx=\int_{1}^{2}\frac{1}{2x}\left((\ln 5)^2-(\ln 1)^2\right)dx=\frac{(\ln 5)^2}{2}\int_{1}^{2}\frac{dx}{x}=\frac{(\ln 5)^2}{2}[\ln x]_1^2$$

$$=\frac{1}{2}(\ln 5)^2\ln 2=0.89773$$

$$\int_{1}^{5}\int_{1}^{2}\frac{\ln y}{xy}dx\,dy$$

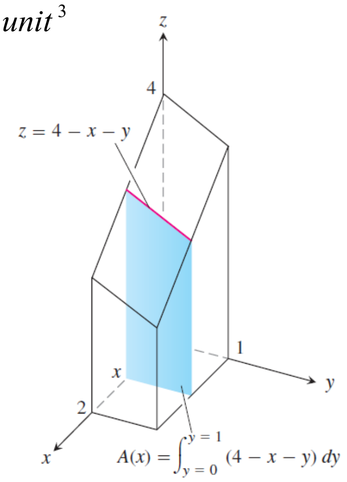
$$=\int_{1}^{5}\frac{\ln y}{y}[\ln x]_1^2 dy=\frac{\ln 2}{2}\left(\left(\ln y\right)^2\right)\Big|_1^5=\frac{\ln 2}{2}\left((\ln 5)^2-(\ln 1)^2\right)==\frac{\ln 2}{2}(\ln 5)^2=0.89773$$

Example [2] Calculate the volume under the plane $z=4-x-y$ over the rectangular region $R:\ 0\le x\le 2,\ 0\le y\le 1$ in the xy-plane.

Solution

$$V=\int_a^b\int_c^d f(x,y)dy\,dx=\int_0^2\int_0^1(4-x-y)dy\,dx=\int_0^2\left(4y-xy-\frac{y^2}{2}\right)\Big|_0^1 dx$$

$$=\int_0^2\left(\frac{7}{2}-x\right)dx=\left(\frac{7}{2}x-\frac{x^2}{2}\right)\Big|_0^2=5unit^3$$

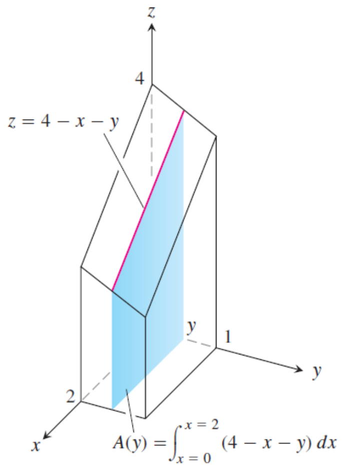


$$V=\int_a^b\int_c^d f(x,y)\,dy\,dx=\int_0^2\int_0^1(4-x-y)\,dy\,dx=\int_0^2\left(4y-xy-\frac{y^2}{2}\right)\Big|_0^1 dx$$

$$=\int_0^2\left(\frac{7}{2}-x\right)dx=\left(\frac{7}{2}x-\frac{x^2}{2}\right)\Big|_0^2=5unit^3$$

$$V=\int_c^d\int_a^b f(x,y)\,dx\,dy=\int_0^1\int_0^2(4-x-y)\,dx\,dy=\int_0^1\left(4x-\frac{2^2}{2}-xy\right)\Big|_0^2 dy$$

$$=\int_0^1(6-2y)\,dy=\left(6y-y^2\right)\Big|_0^1=5unit^3$$

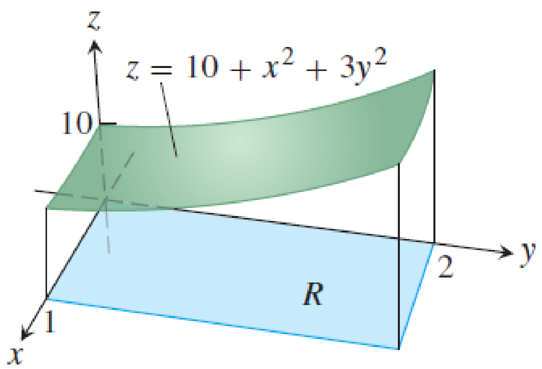


Example [3] Find the volume of the region bounded above by the elliptical $z=10+x^2+3y^2$ and below by the rectangle $R:\ 0\le x\le 1,\ 0\le y\le 2$

Solution

$$V=\int_a^b\int_c^d f(x,y)\,dy\,dx=\int_0^1\int_0^2\left(10+x^2+3y^2\right)dy\,dx=\int_0^1\left(10y+x^2y+y^3\right)\Big|_0^2 dx$$

$$=\int_0^1\left(28+2x^2\right)dx=\left(28x+\frac{2}{3}x^3\right)\Big|_0^1=\frac{86}{3}=28.6667unit^3$$



➢ *Exercises 5-2*

H.W 1- Evaluate the integral:

a) $\int_0^1\int_0^1 \frac{y}{1+xy}\,dx\,dy$

b) $\int_0^1\int_1^2 xye^x\,dy\,dx$

c) $\int_0^{\ln 3}\int_0^{\ln 2} e^{x-y}\,dy\,dx$

d) $\int_0^1\int_0^2 \frac{xy^3}{1+x^2}\,dy\,dx$

e) $\int_0^1\int_0^2 \frac{y}{1+x^2y^2}\,dy\,dx$

f) $\int_0^{-\pi}\int_0^{\pi} y\sin(x+y)\,dy\,dx$

g) $\int_0^1\int_0^1 xy\sqrt{x^2+y^2}\,dy\,dx$

h) $\int_{-1}^1\int_0^1 \frac{xy}{1+x^4}\,dy\,dx$

H.W 2- Find the volume of the solid enclosed by the surface $z=x^2+xy^2$ and the planes $z=0$, $x=0$, $x=5$, $y=2$ and $y=-2$.

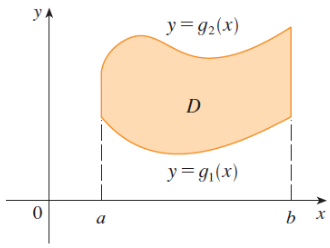
H.W 3- Find the volume of the solid enclosed by the surface $z=1+x^2ye^y$ and the planes $z=0$, $x=1$, $x=-1$, $y=0$, and $y=1$.

H.W 4- Find the volume of the solid in the first octant bounded by the cylinder $z=16-x^2$ and the plane $y=5$.

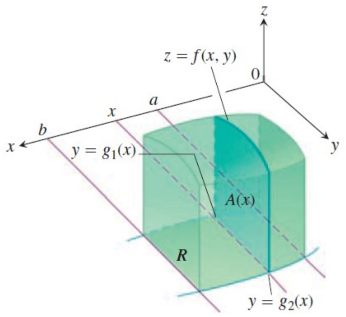
H.W 5- Find the volume of the solid enclosed by the surface $z=2+x^2+(y-2)^2$ and the planes $z=1$, $x=1$, $x=-1$, $y=0$, and $y=4$.

2. Double Integrals over General Regions

If R is a region like the one shown in the xy-plane in figure below, bounded "above" and "below" by the curves $y = g_1(x)$ and $y = g_2(x)$ and on the sides by the lines $x = a, \ y = b$ we may again calculate the volume by :



$$Volume = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$



Example [4] Evaluate the integral :

a) $\int_1^{\ln 8} \int_0^{\ln y} e^{x+y}\,dx\,dy$ b) $\int_1^2 \int_0^{x^2} \frac{1}{x+y}\,dy\,dx$

Solution

Solution : a) $\int_1^{\ln 8} \int_0^{\ln y} e^{x+y}\,dx\,dy$

$$= \int_1^{\ln 8} \int_0^{\ln y} e^{x+y}\,dx\,dy$$

$$= \int_1^{\ln 8} \left(e^{x+y}\right)\Big|_0^{\ln y} dy = \int_1^{\ln 8} \left(e^{\ln y + y} - e^{0+y}\right)dy = \int_1^{\ln 8} \left(ye^y - e^y\right)dy$$

$uv - \int v\,du$

D.		I.
y		e^y
1	↘+	e^y
0	↘−	e^y

$$= \left(\left(ye^y - e^y\right) - e^y\right)\Big|_1^{\ln 8} = \left(ye^y - 2e^y\right)\Big|_1^{\ln 8} = 8\ln 8 + e - 16$$ ⇐ **Ans.**

Solution : b) $\int_1^2 \int_0^{x^2} \frac{1}{x+y}\,dy\,dx$

$$= \int_1^2 \int_0^{x^2} \frac{1}{x+y}\,dy\,dx$$

$$= \int_1^2 \left(\ln(x+y)\right)\Big|_0^{x^2} dx = \int_1^2 \left(\ln\left(x+x^2\right) - \ln(x+0)\right)dx = \int_1^2 \ln\left(\frac{x+x^2}{x}\right)dx = \int_1^2 \ln(1+x)\,dx$$

$$= \left((1+x)\ln(1+x) - (1+x)\right)\Big|_1^2 = \ln\left(\frac{27}{4}\right) - 1$$ ⇐ **Ans.**

Example [5] Evaluate the following integrals by integrating the equivalent reversed order :

a) $\int_0^1\int_y^1 x^2e^{xy}\,dxdy$ b) $\int_0^{\pi}\int_x^{\pi}\frac{\sin y}{y}dydx$ c) $\int_0^1\int_1^{e^x}dydx$ d) $\int_0^2\int_0^{4-x^2}\frac{xe^{2y}}{4-y}dydx$

Solution

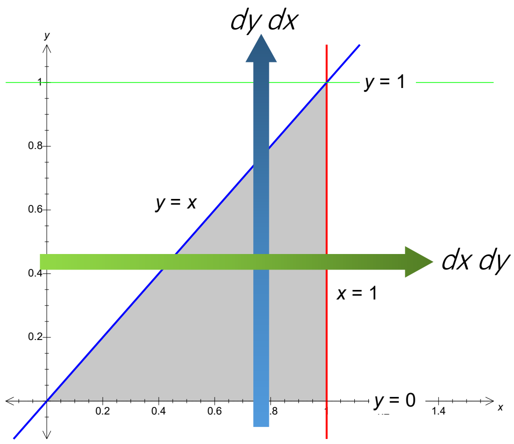
Solution : a) $\int_0^1\int_y^1 x^2e^{xy}\,dxdy$ First , sketch the region of integration

$$=\int_0^1\int_y^1 x^2e^{xy}\,dxdy=\int_0^1\int_0^x x^2e^{xy}\,dydx$$

$$=\int_0^1\int_0^x x\left(xe^{xy}\right)dydx=\int_0^1\left(xe^{xy}\right)\Big|_0^x dx$$

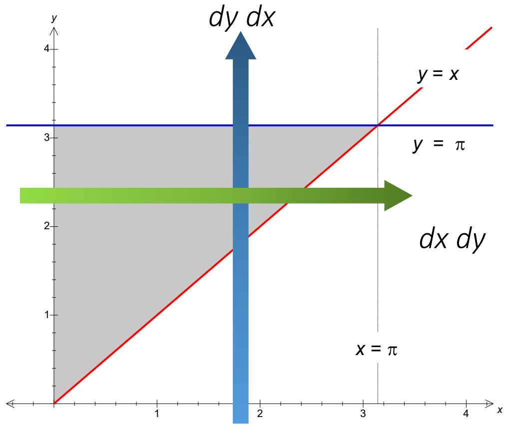
$$=\int_0^1\left(xe^{x^2}-x\right)dx$$

$$=\int_0^1\left(xe^{x^2}-x\right)dx=\frac{e-2}{2}$$ $\Leftarrow$ **Ans.**



Solution : b) $\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\,dydx$

First , sketch the region of integration



$$= \int_0^{\pi}\int_0^{y} \frac{\sin y}{y}\,dxdy$$

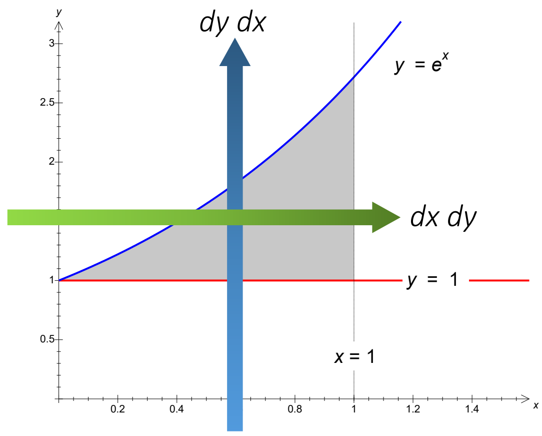
$$= \int_0^{\pi}\left(\frac{\sin y}{y}x\right)\Big|_0^{y} dy$$

$$= \int_0^{\pi}\left(\frac{\sin y}{y}x\right)\Big|_0^{y} dy = \int_0^{\pi}\left(\frac{\sin y}{y}y - \frac{\sin y}{y}0\right)dy$$

$$= \int_0^{\pi}\sin y\,dy = (-\cos y)\Big|_0^{\pi} = ((-\cos\pi)-(-\cos 0)) = 2 \qquad \Leftarrow \textbf{Ans.}$$

Solution : c) $\int_0^{1}\int_1^{e^x} dydx$

First , sketch the region of integration

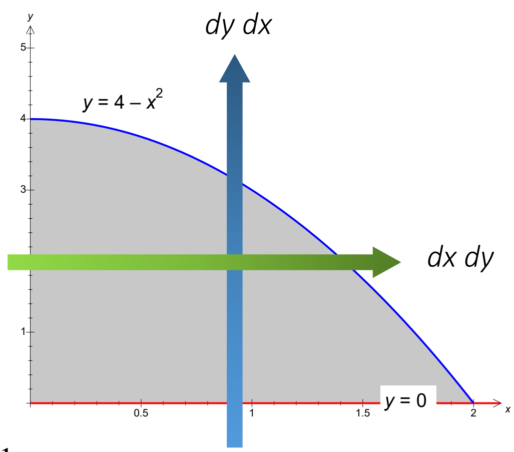


$$= \int_1^{e}\int_{\ln y}^{1} dxdy$$

$$= \int_1^{e}(x)\Big|_{\ln y}^{1} dy$$

$$= \int_1^{e}(1-\ln y)dy$$

$$= (y-(y\ln y - y))\Big|_1^{e} = e-2 \qquad \Leftarrow \textbf{Ans.}$$



Solution : d) $\int_0^{2}\int_0^{4-x^2} \frac{xe^{2y}}{4-y}\,dydx$

First , sketch the region of integration

$$= \int_0^{4}\int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\,dydx$$

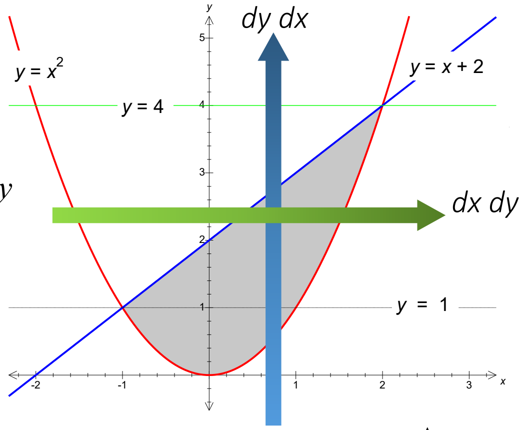
$$= \int_0^{4}\left(\frac{x^2}{2}\frac{e^{2y}}{4-y}\right)\Bigg|_0^{\sqrt{4-y}} dy = \int_0^{4}\left(\frac{e^{2y}}{2}\right)dy = \left(\frac{e^{2y}}{4}\right)\Bigg|_0^{4} = \frac{e^8-1}{4} \qquad \Leftarrow \textbf{Ans.}$$

Example [6] Write and evaluate an equivalent reversed order of integral:

a) $\int_{-1}^{2}\int_{x^2}^{x+2} dydx$ b) $\int_{0}^{1}\int_{\sqrt{y}}^{2-\sqrt{y}} xy\, dxdy$

Solution

Solution : a) $\int_{-1}^{2}\int_{x^2}^{x+2} dydx$ First , sketch the region of integration



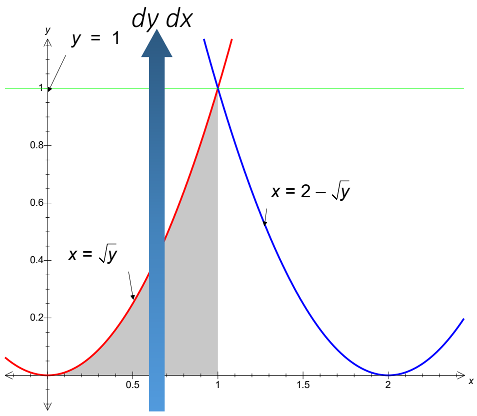
$$= \int_{0}^{1}\int_{-\sqrt{y}}^{+\sqrt{y}} dxdy + \int_{1}^{4}\int_{y-2}^{+\sqrt{y}} dxdy = \int_{0}^{1}(x)\Big|_{-\sqrt{y}}^{+\sqrt{y}} dy + \int_{1}^{4}(x)\Big|_{y-2}^{+\sqrt{y}} dy$$

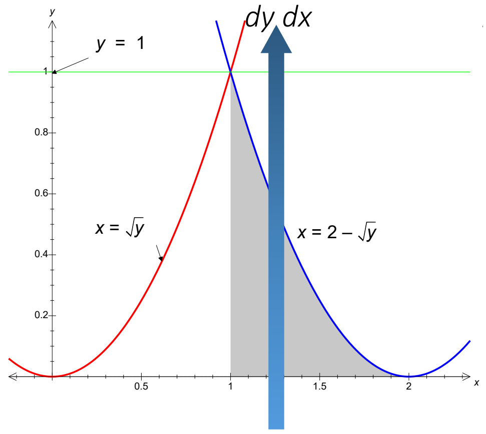
$$= \int_{0}^{1}\left(2\sqrt{y}\right)dy + \int_{1}^{4}\left(\sqrt{y} - y + 2\right)dy$$

$$= \left(\frac{4}{3}\sqrt{y^3}\right)\Bigg|_0^1 + \left(\frac{2}{3}\sqrt{y^3} - \frac{y^2}{2} + 2y\right)\Bigg|_1^4 = 4.5$$

$\Leftarrow$ **Ans.**

Solution : b) $\int_{0}^{1}\int_{\sqrt{y}}^{2-\sqrt{y}} xy\, dxdy$. First , sketch the region of integration





$$= \int_{0}^{1}\int_{0}^{x^2} xy\, dydx + \int_{1}^{2}\int_{0}^{(2-x)^2} xy\, dydx$$

$$= \int_{0}^{1}\left(x\,\frac{y^2}{2}\right)\Bigg|_0^{x^2} dx + \int_{1}^{2}\left(x\,\frac{y^2}{2}\right)\Bigg|_0^{(2-x)^2} dx = \frac{1}{2}\int_{0}^{1}\left(x^5\right)dx + \frac{1}{2}\int_{1}^{2}\left(x\,(2-x)^4\right)dx$$

$$= \frac{1}{2}\left(\frac{x^6}{6}\right)\Bigg|_0^1 + \frac{1}{60}\left((2-x)^5\times(-2-5x)\right)\Big|_1^2 = \frac{1}{12} + \frac{7}{60} = \frac{1}{5}$$

$\Leftarrow$ **Ans.**

Quiz 1 : Evaluate the integral:

a) $\int_{0}^{1}\int_{2y}^{2} 4\cos\left(x^2\right)dxdy$ b) $\int_{0}^{8}\int_{\sqrt[3]{x}}^{2} \frac{1}{1+y^4}\,dydx$ c) $\int_{0}^{4}\int_{0}^{\sqrt{y}} \frac{e^x}{x}\,dxdy$

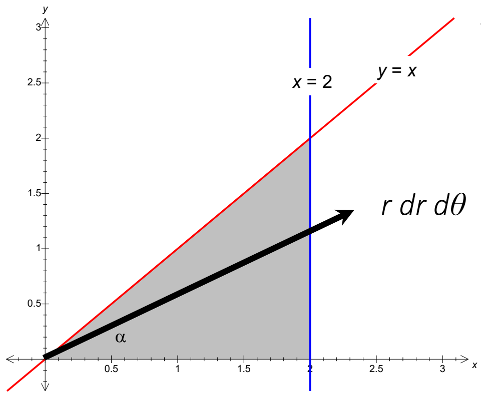
Example [7] Evaluate by using the polar integral:

a) $\int_0^2\int_0^x y\,dydx$ b) $\int_0^2\int_0^{\sqrt{1-(x-1)^2}} \frac{x+y}{x^2+y^2}\,dydx$

Solution

Solution : a) $\int_0^2\int_0^x y\,dydx$

First , sketch the region of integration



$x=2 \Rightarrow r\cos\theta = 2 \Rightarrow r = 2\sec\theta$

$$=\int_0^2\int_0^x y\,dydx = \int_0^{\frac{\pi}{4}}\int_0^{2\sec\theta}(r\sin\theta)(r\,dr\,d\theta)$$

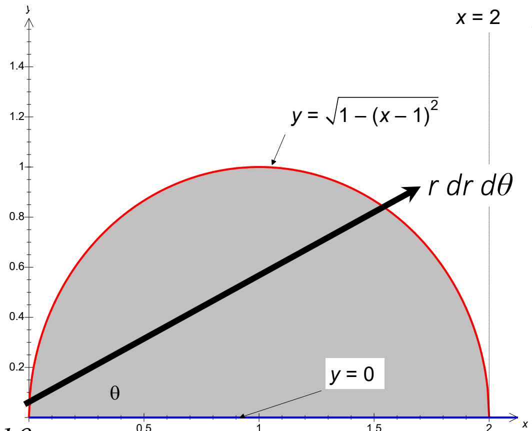
$$=\int_0^{\frac{\pi}{4}}\int_0^{2\sec\theta} r^2\sin\theta\,dr\,d\theta = \int_0^{\frac{\pi}{4}}\left(\frac{r^3}{3}\Bigg|_0^{2\sec\theta}\right)\sin\theta\,d\theta$$

$$=\frac{8}{3}\int_0^{\frac{\pi}{4}}\sec^3\theta\sin\theta\,d\theta$$

$$=\frac{8}{3}\int_0^{\frac{\pi}{4}}\sec^2\theta\tan\theta\,d\theta = \frac{8}{3}\left(\frac{\tan^2\theta}{2}\Bigg|_0^{\frac{\pi}{4}}\right) = \frac{4}{3}$$ $\Leftarrow$ **Ans.**

Solution : b) $\int_0^2\int_0^{\sqrt{1-(x-1)^2}} \frac{x+y}{x^2+y^2}\,dydx$

First , sketch the region of integration



$y=\sqrt{1-(x-1)^2} \Rightarrow y^2+(x-1)^2-1=0$

$\Rightarrow y^2+x^2 = 2x \Rightarrow r^2 = 2r\cos\theta \Rightarrow r = 2\cos\theta$

$$\Rightarrow \int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta}\left(\frac{r\cos\theta + r\sin\theta}{r^2}\right)(r\,dr\,d\theta)$$

$$=\int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta}(\cos\theta+\sin\theta)\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\left(r\Big|_0^{2\cos\theta}\right)(\cos\theta+\sin\theta)\,d\theta$$

$$=\int_0^{\frac{\pi}{2}}(2\cos\theta)(\cos\theta+\sin\theta)\,d\theta = \int_0^{\frac{\pi}{2}}2\left(\cos^2\theta+\cos\theta\sin\theta\right)d\theta$$

$$=2\left[\left(\frac{1}{2}\left(\theta+\frac{\sin2\theta}{2}\right)+\frac{\sin^2\theta}{2}\right)\Bigg|_0^{\frac{\pi}{2}}\right] = \frac{\pi+2}{2}$$ $\Leftarrow$ **Ans.**

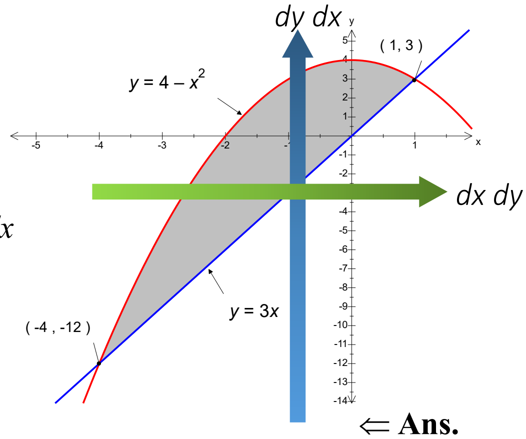
Quiz 2 : Evaluate by using the polar integral:

a) $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{0}\frac{4\sqrt{x^2+y^2}}{1+x^2+y^2}\,dxdy$ b) $\int_0^3\int_0^{\sqrt{3}x}\frac{1}{\sqrt{x^2+y^2}}\,dydx$ c) $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\ln\left(1+x^2+y^2\right)dxdy$

Example [8] Find the volume of the solid whose base is the region in the xy-plane that is bounded by the parabola $y=4-x^2$ and the line $y=3x$ while the top of the solid is bounded by the plane $z=x+4$.

Solution

Solution : First , sketch the region of integration in *xy*-plane



Method (1) : The strip ⊥ x-axis (dydx)

$$=\int_{-4}^{1}\int_{3x}^{4-x^2}(x+4)\,dydx=\int_{-4}^{1}(x+4)y\Big|_{3x}^{4-x^2}dx$$

$$=\int_{-4}^{1}(x+4)\left(4-x^2-3x\right)dx=\int_{-4}^{1}(x+4)\left(4-x^2-3x\right)dx$$

$$=\left(16x-4x^2-\frac{7}{3}x^3-\frac{1}{4}x^4\right)\Big|_{-4}^{1}$$

$$=\frac{625}{12}\,unit^3 \qquad \Leftarrow \textbf{Ans.}$$

Method (2) : The strip ⊥ y-axis (dxdy)

$$=\int_{-12}^{3}\int_{-\sqrt{4-y}}^{\frac{y}{3}}(x+4)\,dxdy+\int_{3}^{4}\int_{-\sqrt{4-y}}^{\sqrt{4-y}}(x+4)\,dxdy$$

$$=\int_{-4}^{1}\left(\frac{1}{2}x^2+4x\right)\Big|_{-\sqrt{4-y}}^{\frac{y}{3}}dy+\int_{3}^{4}\left(\frac{1}{2}x^2+4x\right)\Big|_{-\sqrt{4-y}}^{\sqrt{4-y}}dy$$

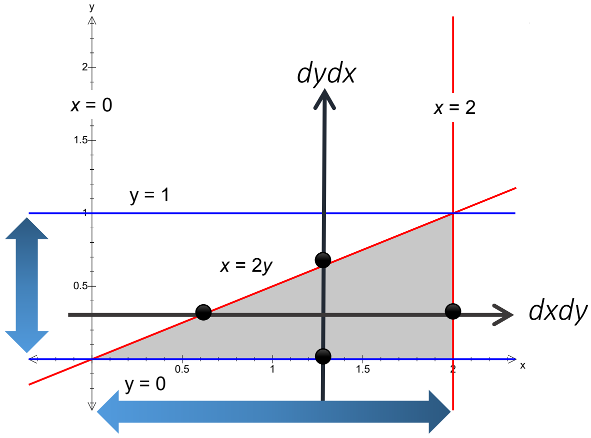
$$=\frac{625}{12}\,unit^3 \qquad \Leftarrow \textbf{Ans.}$$

Quiz 1 : Evaluate the integral:

***Class* A)** $\int_0^1\int_{2y}^2 -8\sin(x^2)\,dx\,dy$ ***Class* B)** $\int_0^1\int_{2y}^2 4\cos(x^2)\,dx\,dy$

Solution : ***Class* A)** $\int_0^1\int_{2y}^2 -8\sin(x^2)\,dx\,dy$

First , sketch the region of integration



$$\int_0^1\int_{2y}^2 -8\sin(x^2)\,dx\,dy = \int_0^2\int_0^{\frac{x}{2}} -8\sin(x^2)\,dy\,dx$$

$$= \int_0^2 -8\sin(x^2)(y)\Big|_0^{\frac{x}{2}} dx = \int_0^2 -8\sin(x^2)\left(\frac{x}{2}-0\right)dx$$

$$= \int_0^2 -4x\sin(x^2)\,dx = \int_0^2 2\left(-2x\sin(x^2)\right)dx$$

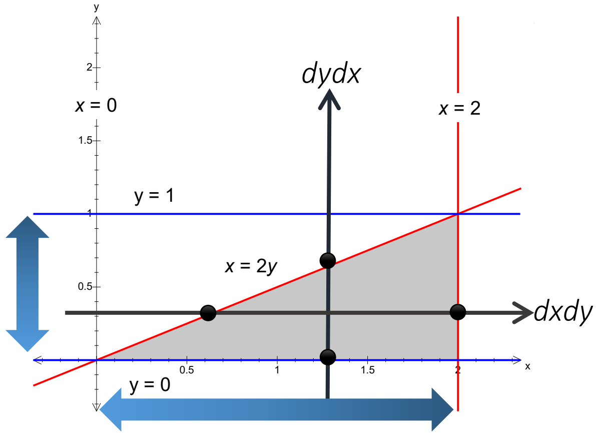
$$= 2\left(\cos(x^2)\right)\Big|_0^2 = 2\left(\cos(2^2)-\cos(0^2)\right)$$

$$= 2\cos(4)-2$$

$\Leftarrow$ **Ans.**

Solution : ***Class* B)** $\int_0^1\int_{2y}^2 4\cos(x^2)\,dx\,dy$

First , sketch the region of integration



$$\int_0^1\int_{2y}^2 4\cos(x^2)\,dx\,dy = \int_0^2\int_0^{\frac{x}{2}} 4\cos(x^2)\,dy\,dx$$

$$= \int_0^2 4\cos(x^2)(y)\Big|_0^{\frac{x}{2}} dx = \int_0^2 4\cos(x^2)\left(\frac{x}{2}-0\right)dx$$

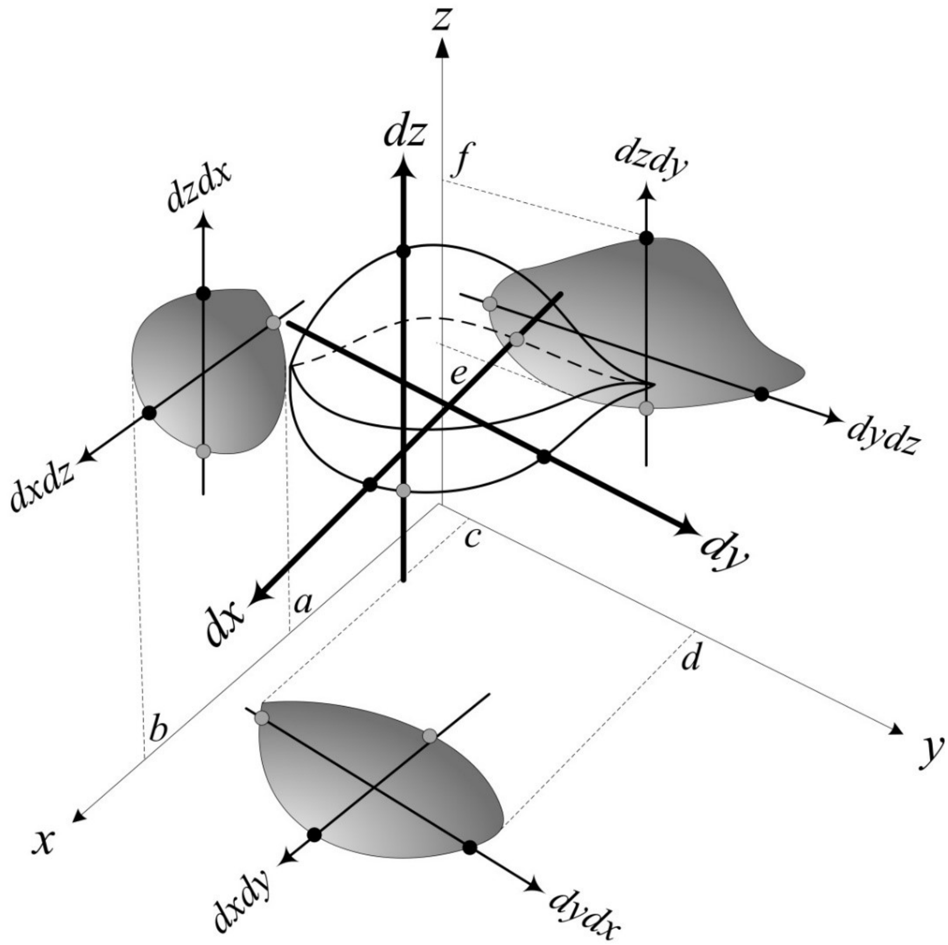
$$= \int_0^2 2x\cos(x^2)\,dx$$

$$= \left(\sin(x^2)\right)\Big|_0^2 = \sin(2^2)-\sin(0^2)$$

$$= \sin(4)$$

$\Leftarrow$ **Ans.**

3. Triple Integrals in rectangular Coordinates



$$I_1=\int_{x=a}^{x=b}\int_{y=f_1(x)}^{y=f_2(x)}\int_{z=h_1(x,y)}^{z=h_2(x,y)} dzdydx$$

$$I_2=\int_{y=c}^{y=d}\int_{x=g_1(y)}^{x=g_2(y)}\int_{z=h_1(x,y)}^{z=h_2(x,y)} dzdxdy$$

$$I_3=\int_{z=e}^{z=f}\int_{x=g_1(z)}^{x=g_1(z)}\int_{y=f_1(x,z)}^{y=f_2(x,z)} dydxdz$$

$$I_4=\int_{x=a}^{x=b}\int_{z=h_1(x)}^{z=h_2(x)}\int_{y=f_1(x,z)}^{y=f_2(x,z)} dydzdx$$

$$I_5=\int_{z=e}^{z=f}\int_{y=f_1(z)}^{y=f_2(z)}\int_{x=g_1(y,z)}^{x=g_2(y,z)} dxdydz$$

$$I_6=\int_{y=c}^{y=d}\int_{z=h_1(y)}^{z=h_2(y)}\int_{x=g_2(y,z)}^{x=g_2(y,z)} dxdzdy$$

Example 9 : Write six different iterated triple integrals for the volume of the tetrahedron cut from the first octant by the plane $6x+3y+2z=6$, and evaluate one of the integrals.

Solution : Sketch and find intersection points

$$I_1=\int_0^1\int_0^{2-2x}\int_0^{\frac{6-6x-3y}{2}} dzdydx$$

$$I_2=\int_0^2\int_0^{\frac{2-y}{2}}\int_0^{\frac{6-6x-3y}{2}} dzdxdy$$

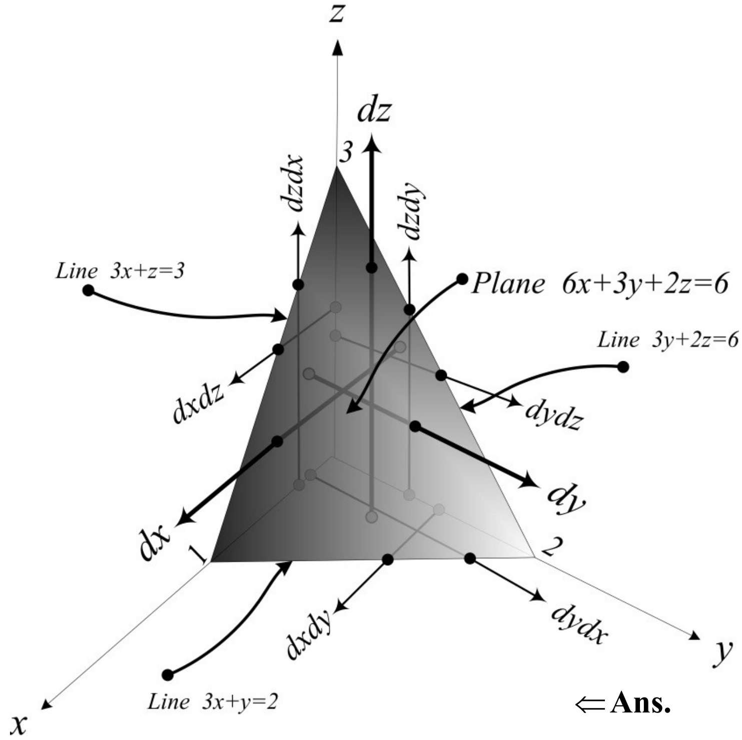
$$I_3=\int_0^1\int_0^{3-3x}\int_0^{\frac{6-6x-2z}{3}} dydzdx$$

$$I_4=\int_0^3\int_0^{\frac{3-z}{3}}\int_0^{\frac{6-6x-2z}{3}} dydxdz$$

$$I_5=\int_0^3\int_0^{\frac{6-2z}{3}}\int_0^{\frac{6-3y-2z}{6}} dxdydz$$

$$I_6=\int_0^2\int_0^{\frac{6-3y}{2}}\int_0^{\frac{6-3y-2z}{6}} dxdzdy$$

$$V=I_1=I_2=I_3=I_4=I_5=I_6=1$$



⇐**Ans.**

Example 10 : Find the **volume** of the a solid bounded below $z=2$ and above by paraboloid $z=6-x^2-y^2$

Solution : Sketch and find region in *xy*-plane

$$z_1=z_2 \Rightarrow 6-x^2-y^2=2$$

$$\Rightarrow x^2+y^2=4 \quad (\text{region } R \text{ in } xy\text{-plane})$$

$$V=\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{+\sqrt{4-y^2}}\int_2^{6-x^2-y^2} dzdxdy$$

$$V=\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{+\sqrt{4-y^2}}\left(4-x^2-y^2\right)dxdy$$

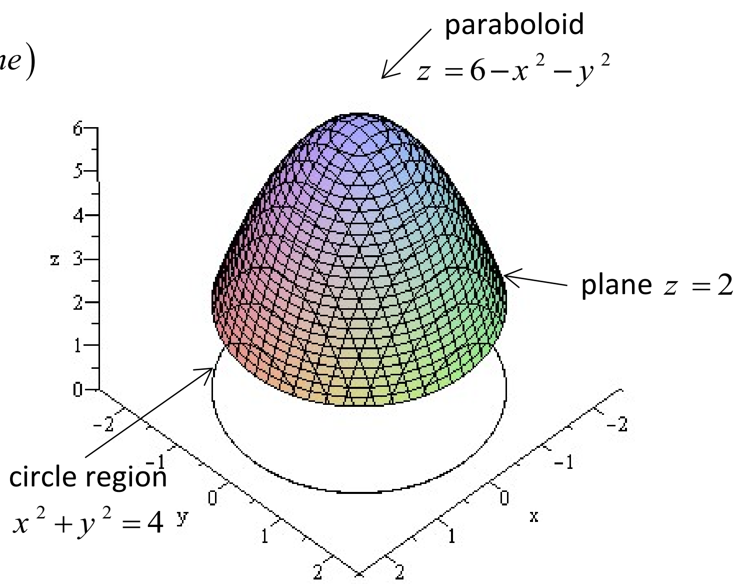
we can used the polar integral

$$V=\int_0^{2\pi}\int_0^2\left(4-r^2\right)rdrd\theta$$

$$V=\int_0^{2\pi}\int_0^2\left(4r-r^3\right)drd\theta$$

$$V=\int_0^{2\pi}\left(2r^2-\frac{r^4}{4}\right)\Bigg|_0^2 d\theta$$

$$V=\int_0^{2\pi}(4)d\theta=8\pi$$



⇐**Ans.**

Example 11 : Write six different iterated triple integrals for the volume of the region in first octant bounded by the planes $x + y = 1$ and $y + 2z = 1$, and evaluate the volume .

Solution : Sketch and find intersection points

$$z_1 = z_2 \quad \Rightarrow 1 - x = \frac{2-y}{2} \quad \Rightarrow \quad y = 2x$$

$$I_1 = \int_0^1 \int_0^{2-2z} \int_0^{1-z} dxdydz$$

$$I_2 = \int_0^2 \int_0^{\frac{2-y}{2}} \int_0^{1-z} dxdzdy$$

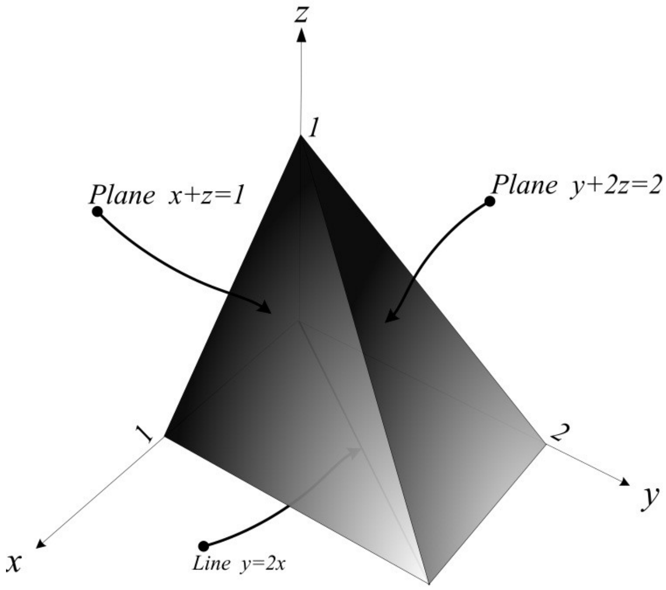
$$I_3 = \int_0^1 \int_0^{1-z} \int_0^{2-2z} dydxdz$$

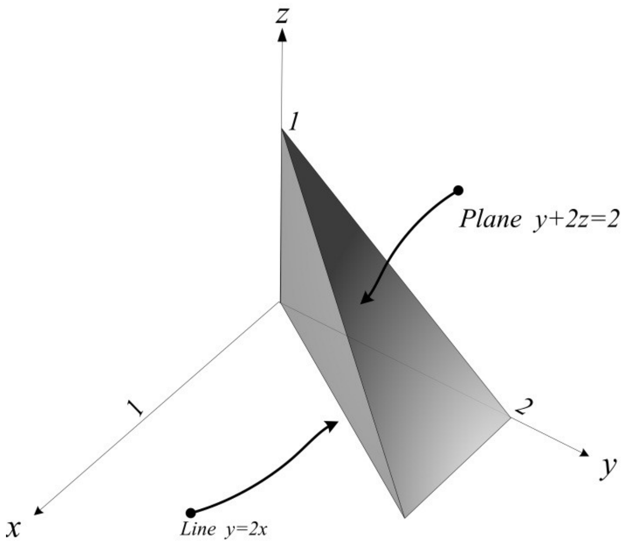
$$I_4 = \int_0^1 \int_0^{1-x} \int_0^{2-2z} dydzdx$$

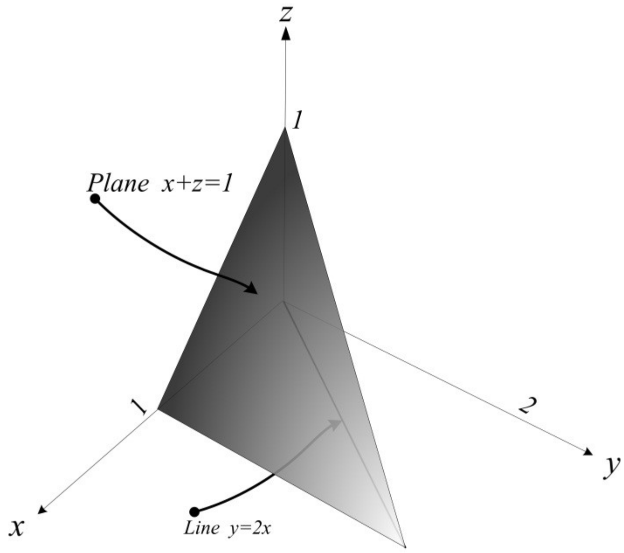
$$I_5 = \int_0^1 \int_{2x}^2 \int_0^{\frac{2-y}{2}} dzdydx + \int_0^1 \int_0^{2x} \int_0^{1-x} dzdydx$$

$$I_6 = \int_0^2 \int_0^{\frac{y}{2}} \int_0^{\frac{2-y}{2}} dzdxdy + \int_0^2 \int_{\frac{y}{2}}^1 \int_0^{1-x} dzdxdy$$

$$V = I_1 = I_2 = I_3 = I_4 = I_5 = I_6 = \frac{2}{3}$$ $\Leftarrow$ **Ans.**





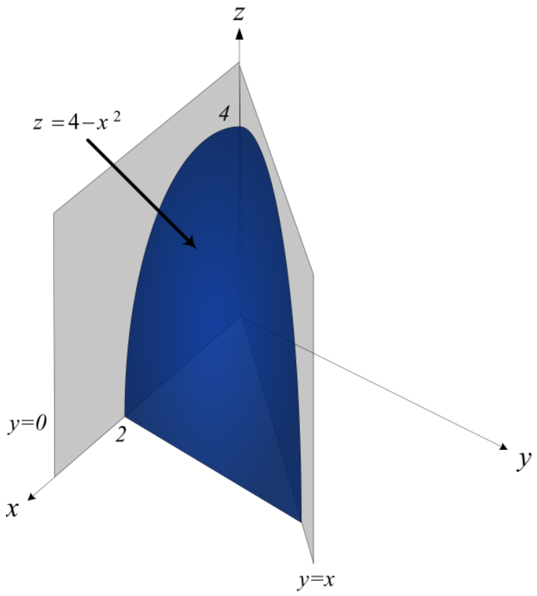


Example 12: Evaluate the integral:

$$\int_0^2\int_0^x\int_0^{4-x^2}\frac{\sin(2z)}{4-z}dzdydx$$

Solution : We cannot solve this triple integral by direct method. Therefore, we will sketched and find suitable region for the triple integral.

$$\int_0^2\int_0^x\int_0^{4-x^2}\frac{\sin(2z)}{4-z}dzdydx$$



Method (1)

$$\int_0^4\int_0^{\sqrt{4-z}}\int_0^x\frac{\sin(2z)}{4-z}dydxdz$$

$$=\int_0^4\int_0^{\sqrt{4-z}}\frac{\sin(2z)}{4-z}(y)\Big|_0^x dxdz$$

$$=\int_0^4\int_0^{\sqrt{4-z}}\frac{\sin(2z)}{4-z}(x)dxdz$$

$$=\int_0^4\frac{\sin(2z)}{4-z}\left(\frac{x^2}{2}\right)\Bigg|_0^{\sqrt{4-z}}dz$$

$$=\frac{1}{2}\int_0^4\frac{\sin(2z)}{4-z}\left(\sqrt{4-z}\right)^2dz=\frac{1}{2}\int_0^4\frac{\sin(2z)}{4-z}(4-z)dz$$

$$=\frac{1}{2}\int_0^4\sin(2z)dz=\frac{-1}{4}\int_0^4-2\sin(2z)dz=\frac{-1}{4}(\cos(2z))\Big|_0^4=\frac{1}{4}(1-\cos(8))$$ $\Leftarrow$ **Ans.**

Method (2)

$$\int_0^4\int_0^{\sqrt{4-z}}\int_y^{\sqrt{4-z}}\frac{\sin(2z)}{4-z}dxdydz$$

$$=\int_0^4\int_0^{\sqrt{4-z}}\frac{\sin(2z)}{4-z}(x)\Big|_y^{\sqrt{4-z}}dydz$$

$$=\int_0^4\int_0^{\sqrt{4-z}}\frac{\sin(2z)}{4-z}\left(\sqrt{4-z}-y\right)dydz$$

$$=\int_0^4\frac{\sin(2z)}{4-z}\left(y\sqrt{4-z}-\frac{y^2}{2}\right)\Bigg|_0^{\sqrt{4-z}}dz$$

$$=\int_0^4\frac{\sin(2z)}{4-z}\left(\sqrt{4-z}\sqrt{4-z}-\frac{\left(\sqrt{4-z}\right)^2}{2}\right)dz=\int_0^4\frac{\sin(2z)}{(4-z)}\frac{(4-z)}{2}dz$$

$$=\frac{1}{2}\int_0^4\sin(2z)dz=\frac{-1}{4}\int_0^4-2\sin(2z)dz=\frac{-1}{4}(\cos(2z))\Big|_0^4=\frac{1}{4}(1-\cos(8))$$ $\Leftarrow$ **Ans.**

4. Refrence

Calculus, Eighth Edition James Stewart Page 1077

Link : https://www.amazon.com/Calculus-MindTap-Course-James-Stewart-ebook-dp-B00YHKU50E/dp/B00YHKU50E/ref=mt_other?_encoding=UTF8&me=&qid=

Chapter Seven

Differential Equation

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1- Differential Equation

A ***differential equation*** is an equation involving an unknown function and its derivatives. A differential equation is an ***ordinary differential equation*** (ODE) if the unknown function depends on only one independent variable. If the unknown function depends on two or more independent variables, the differential equation is a ***partial differential equation*** (PDE).

Order : is the highest derivative in the D.E. ***Degree*** : is the highest exponent of an order.

Example: The following are differential equations involving the unknown function y.

$$\frac{dy}{dx} = 9x - 4$$

$$e^{y}\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 1$$

$$4\frac{d^3y}{dx^3} + \sin x\left(\frac{dy}{dx}\right)^7 + y^3\left(\frac{dy}{dx}\right)^2 = 5x$$

$$\frac{\partial^2 y}{\partial t^2} - 4\frac{\partial^2 y}{\partial x^2} = 0$$

Expression		Used to represent
y'	$\frac{dy}{dx}$	First derivative
y''	$\frac{d^2y}{dx^2}$	Second derivative
y'''	$\frac{d^3y}{dx^3}$	Third derivative
$y^{(4)}$	$\frac{d^4y}{dx^4}$	Forth derivative
$y^{(n)}$	$\frac{d^ny}{dx^n}$	*n*th derivative
$(y')^m$	$\left(\frac{dy}{dx}\right)^m$	(First derivative)^m
$(y^{(5)})^9$	$\left(\frac{d^5y}{dx^5}\right)^9$	(Fifth derivative)^9
$(y^{(n)})^m$	$\left(\frac{d^ny}{dx^n}\right)^m$	(*n*th derivative)^m
$(y^9)^{(5)}$	$\frac{d^5}{dx^5}(y^9)$	Fifth derivative (function)^m
$(y^m)^{(n)}$	$\frac{d^n}{dx^n}(y^m)$	*n*th derivative (function)^m

2- First Order Differential Equations

Standard form for a First-order differential equation in the unknown function y(x) is:

$$y' = f(x, y)$$

First Order Differential Equation Types:

1. Separable Equations
2. Homogeneous Equations
3. Exact Equations
4. Linear Equations
5. Bernoulli Equations

2.1- Separable Differential Equations

Procedure to solve a Separable First Order Differential Equation

1- Write the equation in the form :

$$\boldsymbol{A(x)dx + B(y)dy} = 0$$

2- Integrate A(*x*) with respect to *x* and B(*y*) with respect to *y* to obtain an equation that relates y and x .

Example 1 : Solve differential equations

(a) $y(1+x^2)dy = dx$ **(b)** $dx + xydy + y^2dx + ydy = 0$

(c) $\frac{1}{y}dx - \frac{x}{y^2}dy = 0$ **(d)** $\frac{dy}{dx} = e^{3x+2y} - 4$

(e) $e^{x+2y}dy - e^{y-2x}dx = 0$ **(f)** $(y \ln y)dx + (1+x^2)dy = 0 \quad y(0) = e$

Solution : (a) $y(1+x^2)dy = dx$

$$y(1+x^2)dy = dx$$

$$ydy = \frac{dx}{1+x^2}$$

$$\frac{dx}{1+x^2} - ydy = 0$$

$$\int \frac{1}{1+x^2}dx - \int ydy = C \quad \Rightarrow \quad \tan^{-1}x - \frac{y^2}{2} = C \quad \Leftarrow \textbf{Ans.}$$

Solution : (b) $dx + xydy + y^2dx + ydy = 0$

Solution : (c) $\frac{1}{y}dx - \frac{x}{y^2}dy = 0$

$\frac{1}{y}dx - \frac{x}{y^2}dy = 0 \quad \Rightarrow \quad \frac{1}{y}dx = \frac{x}{y^2}dy \quad \Rightarrow \quad \frac{1}{x}dx = \frac{y}{y^2}dy$

$\frac{1}{x}dx - \frac{1}{y}dy = 0$

$\int \frac{1}{x}dx - \int \frac{1}{y}dy = C \quad \Rightarrow \quad \ln x - \ln y = C$

$y = C_1 x$ $\Leftarrow$ **Ans.**

Solution : (d) $\frac{dy}{dx} = e^{3x+2y} + 4$

Solution : (e) $e^{x+2y}dy - e^{y-2x}dx = 0$

$$e^{x+2y}dy - e^{y-2x}dx = 0$$

$$\frac{dy}{dx} = \frac{e^{y-2x}}{e^{x+2y}}$$

$$\frac{dy}{dx} = e^{(y-2x)-(x+2y)} \quad \Rightarrow \quad \frac{dy}{dx} = e^{(y-2y)-(x+2x)}$$

$$\frac{dy}{dx} = e^{-y-3x}e^{y} \quad \Rightarrow \quad dy = e^{-3x}dx$$

$$-e^{-3x}dx + e^{y}dy = 0$$

$$\int -e^{-3x}dx + \int e^{y}dy = C$$

$$\frac{e^{-3x}}{3} + e^{y} = C$$

$$y = ln\left(C - \frac{e^{-3x}}{3}\right) \quad \Leftarrow \textbf{Ans.}$$

Solution : (f) $(y\ ln\ y)dx + (1+x^2)dy = 0 \quad y(0) = e$

$$(y\ ln\ y)dx + (1+x^2)dy = 0$$

$$(y\ ln\ y)dx = -(1+x^2)dy \quad \Rightarrow \quad \frac{1}{(1+x^2)}dx = \frac{-1}{(y\ ln\ y)}dy$$

$$\frac{1}{(1+x^2)}dx + \frac{1}{(y\ ln\ y)}dy = 0$$

$$\int \frac{1}{(1+x^2)}dx + \int \frac{1}{(y\ ln\ y)}dy = C \quad \Rightarrow \quad tan^{-1}(x) + ln(ln\ y) = C$$

$$y(0) = e \quad \Rightarrow \quad tan^{-1}(0) + ln(ln\ e) = C$$

$$0 + ln(1) = C \quad \Rightarrow \quad C = 0$$

$$tan^{-1}(x) + ln(ln\ y) = 0$$

$$y = e^{e^{-tan^{-1}(x)}} \quad \Leftarrow \textbf{Ans.}$$

H.W 1 : Solve differential equations

(a) $\frac{dy}{dx} = \frac{e^{2x+y}}{e^{x-y}}$ (b) $\sqrt{1+(y')^2} = ky \quad (k\ \ is\ constant)$

(c) $x^2 y\frac{dy}{dx} = (1+x)\ csc\ y$ (d) $xe^{y}dy + \frac{1+x^2}{y}dx = 0$

(e) $x(2y-3)dx + (1+x^2)dy = 0$

2.2- Homogenous Differential Equations

Certain first order differential equations are not of the 'variable-separable' type, but can be made separable by changing the variable.

An equation of the form

$$N(x,y)\frac{dy}{dx}=M(x,y)$$

Where N and M are functions of both x and y of the ***same degree*** throughout, is said to be ***Homogeneous*** in y and x.

Homogenous First Order Differential Equation

A first order differential equation is *Homogenous* if it can be put in the form:

$$\frac{dy}{dx}=F\left(\frac{y}{x}\right) \qquad (2)$$

or

$$\frac{dx}{dy}=F\left(\frac{x}{y}\right) \qquad (3)$$

$$Let\ \ v=\frac{y}{x} \quad \Rightarrow \quad y=xv$$

$$\therefore \frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$\because \frac{dy}{dx}=F\left(\frac{y}{x}\right)=F(v)$$

$$\therefore v+x\frac{dv}{dx}=F(v)$$

$$\therefore \frac{dx}{x}+\frac{dv}{v-F(v)}=0$$

$$\therefore \int\frac{dx}{x}+\int\frac{dv}{v-F(v)}=C$$

$$\boxed{\int\frac{dv}{v-F(v)}=C-\ln|x|}$$

$$Let\ \ u=\frac{x}{y} \quad \Rightarrow \quad x=yu$$

$$\therefore \frac{dx}{dy}=u+y\frac{du}{dy}$$

$$\because \frac{dx}{dy}=F\left(\frac{x}{y}\right)=F(u)$$

$$\therefore u+y\frac{du}{dy}=F(u)$$

$$\therefore \frac{dy}{y}+\frac{du}{u-F(u)}=0$$

$$\therefore \int\frac{dy}{y}+\int\frac{du}{u-F(u)}=C$$

$$\boxed{\int\frac{du}{u-F(u)}=C-\ln|y|}$$

Procedure to solve a Homogenous First Order Differential Equation

1. Rewrite $N(x,y)\dfrac{dy}{dx} = M(x,y)$ into the form $\dfrac{dy}{dx} = \dfrac{M(x,y)}{N(x,y)}$
2. Make the substitution $v = \dfrac{y}{x}$.
3. Rewrite $\dfrac{M(x,y)}{N(x,y)}$ into the form $F\left(\dfrac{y}{x}\right) = F(v)$.
4. Solve $\displaystyle\int \frac{dv}{v - F(v)} = C - \ln|x|$
5. Rewrite the solution into the form of x and y ($v = \dfrac{y}{x}$).

<u>Note</u>:

If the form of a Homogenous (F-ODE) is $\dfrac{dx}{dy} = \dfrac{M(x,y)}{N(x,y)}$, the steps are:

1. Rewrite $N(x,y)\dfrac{dy}{dx} = M(x,y)$ into the form $\dfrac{dx}{dy} = \dfrac{M(x,y)}{N(x,y)}$
2. Make the substitution $u = \dfrac{x}{y}$.
3. Rewrite $\dfrac{M(x,y)}{N(x,y)}$ into the form $F\left(\dfrac{x}{y}\right) = F(u)$.
4. Solve $\displaystyle\int \frac{du}{u - F(u)} = C - \ln|y|$
5. Rewrite the solution into the form of x and y ($u = \dfrac{x}{y}$).

Example 2 : Solve differential equations

(a) $\dfrac{dx}{x-y} = \dfrac{dy}{x+y}$ (b) $\dfrac{dx}{xy} = \dfrac{dy}{x^2+y^2}$

(c) $\dfrac{dy}{xy} = \dfrac{dx}{x^2+y^2}$ (d) $\left(xe^{\frac{y}{x}} + y\right)dx - xdy = 0$

(e) $\left(x\sin\left(\dfrac{y}{x}\right) - y\cos\left(\dfrac{y}{x}\right)\right)dx + \left(x\cos\left(\dfrac{y}{x}\right)\right)dy = 0 \quad , \; y(2) = \pi$

(f) $\dfrac{dy}{dx} = \dfrac{4x+6y+1}{2x-3y}$ (g) $\dfrac{dy}{dx} = \dfrac{x+y-2}{x-y+3}$

Solution : (a) $\dfrac{dx}{x-y}=\dfrac{dy}{x+y}$

Solution : (b) $\dfrac{dx}{xy}=\dfrac{dy}{x^2+y^2}$

$$\frac{dx}{xy}=\frac{dy}{x^2+y^2}$$

$$\frac{x^2+y^2}{xy}=\frac{dy}{dx} \qquad \Rightarrow \qquad \frac{dy}{dx}=\frac{x^2}{xy}+\frac{y^2}{xy}$$

$$\frac{dy}{dx}=\frac{x}{y}+\frac{y}{x}$$

$$Let \quad v=\frac{y}{x} \Rightarrow F(v)=\frac{1}{v}+v$$

$$\because \int\frac{dx}{x}+\int\frac{dv}{v-F(v)}=C \qquad \Rightarrow \qquad \int\frac{dx}{x}+\int\frac{dv}{v-\left(\frac{1}{v}+v\right)}=C$$

$$\int\frac{dx}{x}-\int v\,dv=C \qquad \Rightarrow \qquad \ln x-\frac{v^2}{2}=C \qquad \Rightarrow \qquad \ln x-\frac{\left(\frac{y}{x}\right)^2}{2}=C$$

$$y=\pm x\sqrt{2\ln x+C_1} \quad (C_1=2C)$$ $\Leftarrow$ **Ans.**

Solution : (c) $\dfrac{dy}{xy}=\dfrac{dx}{x^2+y^2}$

$$\frac{dy}{xy}=\frac{dx}{x^2+y^2}$$

$$\frac{x^2+y^2}{xy}=\frac{dx}{dy} \qquad \Rightarrow \qquad \frac{dx}{dy}=\frac{x^2}{xy}+\frac{y^2}{xy}$$

$$\frac{dx}{dy}=\frac{x}{y}+\frac{y}{x}$$

$$Let \quad u=\frac{y}{x} \Rightarrow F(u)=\frac{1}{u}+u$$

$$\because \int\frac{dy}{y}+\int\frac{du}{u-F(u)}=C \qquad \Rightarrow \qquad \int\frac{dy}{y}+\int\frac{du}{u-\left(\frac{1}{u}+u\right)}=C$$

$$\int\frac{dy}{y}-\int u\,du=C \qquad \Rightarrow \qquad \ln y-\frac{u^2}{2}=C \qquad \Rightarrow \qquad \ln y-\frac{\left(\frac{x}{y}\right)^2}{2}=C$$

$$2\ln y-\left(\frac{x}{y}\right)^2=2C \qquad \Leftarrow \textbf{Ans.}$$

Solution : (d) $\left(xe^{\frac{y}{x}}+y\right)dx-xdy=0$

$$\left(xe^{\frac{y}{x}}+y\right)dx-xdy=0 \qquad \Rightarrow \frac{dy}{dx}=\frac{xe^{\frac{y}{x}}+y}{x} \qquad \Rightarrow \frac{dy}{dx}=\frac{xe^{\frac{y}{x}}}{x}+\frac{y}{x} \qquad \Rightarrow \frac{dy}{dx}=e^{\frac{y}{x}}+\frac{y}{x}$$

$$Let \quad v=\frac{y}{x} \Rightarrow F(v)=e^v+v$$

$$\because \int\frac{dx}{x}+\int\frac{dv}{v-F(v)}=C \qquad \Rightarrow \qquad \int\frac{dx}{x}+\int\frac{dv}{v-(e^v+v)}=C$$

$$\therefore \int\frac{dx}{x}-\int e^{-v}dv=C \qquad \Rightarrow \qquad \ln(x)+e^{-v}=C \qquad \Rightarrow \qquad \ln(x)+e^{-\frac{y}{x}}=C$$

$$y=-x\ln\left(C-\ln(x)\right) \qquad \Leftarrow \textbf{Ans.}$$

Solution : (e) $\left(x\sin\left(\frac{y}{x}\right)-y\cos\left(\frac{y}{x}\right)\right)dx+\left(x\cos\left(\frac{y}{x}\right)\right)dy=0 \quad , y(2)=\pi$

$$\left(\frac{x\sin\left(\frac{y}{x}\right)}{x\cos\left(\frac{y}{x}\right)}-\frac{y\cos\left(\frac{y}{x}\right)}{x\cos\left(\frac{y}{x}\right)}\right)dx+\left(\frac{x\cos\left(\frac{y}{x}\right)}{x\cos\left(\frac{y}{x}\right)}\right)dy=0\div x\cos\left(\frac{y}{x}\right)$$

$$\left(\tan\left(\frac{y}{x}\right)-\frac{y}{x}\right)dx+dy=0 \quad \Rightarrow \frac{dy}{dx}=\frac{y}{x}+\tan\left(\frac{y}{x}\right)$$

$$Let \quad v=\frac{y}{x} \Rightarrow F(v)=v+\tan(v)$$

$$\because \int\frac{dx}{x}+\int\frac{dv}{v-F(v)}=C \quad \Rightarrow \quad \int\frac{dx}{x}+\int\frac{dv}{v-(v+\tan(v))}=C$$

$$\int\frac{dx}{x}-\int\cot(v)dv=C \quad \Rightarrow \quad \ln x-\ln(\sin(v))=C \quad \Rightarrow \quad y=x\sin^{-1}\left(\frac{x}{C}\right)$$

$$y(2)=\pi \quad \Rightarrow \quad \pi=2\sin^{-1}\left(\frac{2}{C}\right) \quad \Rightarrow \quad C=2$$

$$y=x\sin^{-1}\left(\frac{x}{2}\right)$$ $\Leftarrow$ **Ans.**

Solution : (f) $\frac{dy}{dx}=\frac{4x+6y+1}{2x-3y}$

Solution : (g) $\frac{dy}{dx}=\frac{x+y-2}{x-y+3}$

H.W 2 : Solve differential equations

(a) $\frac{dy}{dx}=\frac{\sqrt{x^2+y^2}-x}{y}$

(b) $x\frac{dy}{dx}=y+\sqrt{x^2+y^2}$

(c) $\left(x^2+y^2\right)dy-y^2dx=0$

(d) $\frac{dy}{dx}=\frac{e^y-e^x}{e^y+e^x}+1$

(e) $\frac{dy}{dx}=\frac{2x-y}{x-2y}$

(f) $x\frac{dy}{dx}-y=\sqrt{x^2-y^2}$

(g) $\frac{dy}{dx}=\frac{x+y}{x-y+2}$

(h) $\frac{dy}{dx}=\frac{5x-3y+7}{5x-3y+1}$

(i) $y'=\frac{y\left(1+ln\left(y\right)-ln\left(x\right)\right)}{x\left(ln\left(y\right)-ln\left(x\right)\right)}$, $y\left(1\right)=1$

2.3- Exact Differential Equations

A first order differential equation is ***Exact*** if it can be put in the form:

$$M(x,y)dx + N(x,y)dy = 0 \qquad (4)$$

or

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \qquad (5)$$

$$\therefore M = \frac{\partial f}{\partial x} \quad , \quad \therefore N = \frac{\partial f}{\partial y}$$

And having the property that

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Procedure to solve a Exact Differential Equation

1. Rewrite the equation in the form :
$$M(x,y)dx + N(x,y)dy = 0$$
2. Integrate $M(x,y)$ with respect to x, writing the constant of integration as $k(y)$
$$f(x,y) = \int_{y\ constant} M(x,y)\,dx + k(y) \qquad (6)$$
3. Differentiate with respect to y, and set result equal $N(x,y)$ to find $k'(y)$
$$N(x,y) = \frac{1}{\partial y}\left(\int_{y\ constant} M(x,y)\,dx\right) + k'(y)$$
4. Integrate to find $k(y)$ and substituted in Eq.(6), then writing the solution of exact equation as

__General Solution__

$$R = \int_{y-constant} M(x,y)\,dx \qquad\qquad K = \int_{x-constant}\left(N - \frac{\partial R}{\partial y}\right)dy = \int N(0,y)\,dy$$

$$\boxed{R + K = C}$$

Example 3 : Solve differential equations

(a) $2xydx+\left(1+x^2\right)dy=0$

(b) $\left(y^2-1\right)dx+\left(\sin y-2xy\right)dy=0$

(c) $\left(e^x+\ln(y)+\frac{y}{x}\right)dx+\left(\frac{x}{y}+\ln(x)+\sin(y)\right)dy=0$

(d) $\left(x+\sqrt{1+y^2}\right)dx-\left(y-\frac{xy}{\sqrt{1+y^2}}\right)dy=0$

Solution : (a) $2xydx+\left(1+x^2\right)dy=0$

Solution : (b) $\left(y^2-1\right)dx+\left(\sin y-2xy\right)dy=0$

Solution : (c) $\left(e^x+\ln(y)+\dfrac{y}{x}\right)dx+\left(\dfrac{x}{y}+\ln(x)+\sin(y)\right)dy=0$

$$M=e^x+\ln(y)+\frac{y}{x} \quad \& \quad N=\frac{x}{y}+\ln(x)+\sin(y)$$

$$\frac{\partial M}{\partial y}=\frac{1}{y}+\frac{1}{x} \quad \& \quad \frac{\partial N}{\partial x}=\frac{1}{y}+\frac{1}{x} \quad (\therefore exact)$$

$$f(x,y)=\int_{y\ constant}\left(e^x+\ln(y)+\frac{y}{x}\right)dx+k(y)$$

$$f(x,y)=e^x+x\ln(y)+y\ln(x)+k(y)$$

$$N=\frac{\partial f}{\partial y} \Rightarrow \frac{x}{y}+\ln(x)+\sin(y)=\frac{x}{y}+\ln(x)+k'(y) \Rightarrow k'(y)=\sin(y)$$

$$\therefore k(y)=-\cos(y)$$

$$\therefore e^x+x\ln(y)+y\ln(x)-\cos(y)=C$$ $\Leftarrow$ **Ans.**

Solution : (d) $\left(x+\sqrt{1+y^2}\right)dx-\left(y-\dfrac{xy}{\sqrt{1+y^2}}\right)dy=0$

$$M=x+\sqrt{1+y^2} \quad \& \quad N=y-\frac{xy}{\sqrt{1+y^2}}$$

$$\frac{\partial M}{\partial y}=-\frac{y}{\sqrt{1+y^2}} \quad \& \quad \frac{\partial N}{\partial x}=-\frac{y}{\sqrt{1+y^2}} \quad (\therefore Exact)$$

$$R=\int_{y-constant} M\,dx=\int_{y-constant}\left(x+\sqrt{1+y^2}\right)dx=\frac{x^2}{2}+x\sqrt{1+y^2}$$

$$\frac{\partial R}{\partial y}=-\frac{xy}{\sqrt{1+y^2}}$$

$$K=\int_{x-constant}\left(N-\frac{\partial R}{\partial y}\right)dy=\int_{x-constant}\left(y-\frac{xy}{\sqrt{1+y^2}}-\left(-\frac{xy}{\sqrt{1+y^2}}\right)\right)dy=\int_{x-constant} y\,dy=\frac{y^2}{2}$$

$$R+K=C \Rightarrow \frac{x^2}{2}+x\sqrt{1+y^2}+\frac{y^2}{2}=C$$ $\Leftarrow$ **Ans.**

H.W 3 : Solve differential equation

(a) $\left(e^x+\ln(y)\right)dx+\left(\dfrac{x}{y}+1\right)dy=0\ ,\ y(\ln(2))=1$

(b) $\left(\dfrac{y^2}{1+x^2}-2y\right)dx+\left(2y\tan^{-1}(x)-2x+\sinh(y)\right)dy=0$

(c) $\left(x+\sqrt{1+y^2}\right)dx-\left(y-\dfrac{xy}{\sqrt{1+y^2}}\right)dy=0$

(d) $\left(\sin(x)+\tan^{-1}\left(\dfrac{y}{x}\right)\right)dx-\left(y-\ln\left(\sqrt{x^2+y^2}\right)\right)dy=0$

2.3.1 Integration factor

It can be shown that every ***nonexact*** $\left(\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\right)$ differential equation, can be made exact by multiplying both sides by a suitable ***integrating factor*** $I(x,y)$.

$$I(x,y)M(x,y)dx + I(x,y)N(x,y)dy = 0$$

	Condition	Integrating factor
Case [1]	If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(x)$	$I(x,y) = e^{\int g(x)dx}$
Case [2]	If $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = h(y)$	$I(x,y) = e^{-\int h(y)dy}$
Case [3]	If $M = y\,f(xy)$ and $N = x\,g(xy)$	$I(x,y) = \frac{1}{xM - yN}$

Example 4 : Solve differential equations

(a) $ydx - xdy = 0$ (b) $(x+3y)dx + xdy = 0$ (c) $2xy^2dx + 3x^2ydy = 0$

Solution : (a) $ydx - xdy = 0$

$M = y$ & $N = -x$ $\frac{\partial M}{\partial y} = 1$ & $\frac{\partial N}{\partial x} = -1$ $(\therefore\ nonexact)$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(x) = \frac{1}{-x}(1-(-1)) = -\frac{2}{x}$$

$$I = e^{\int g(x)dx} = e^{\int -\frac{2}{x}dx} = \frac{1}{x^2}$$

$$Integtation\ factor\ I = \frac{1}{x^2}$$

$$(IM)dx + (IN)dy = 0 \quad \Rightarrow \quad \left(\frac{y}{x^2}\right)dx + \left(-\frac{1}{x}\right)dy = 0$$

$$\frac{\partial(IM)}{\partial y} = \frac{1}{x^2} \quad \& \quad \frac{\partial(IN)}{\partial x} = \frac{1}{x^2} \quad (\therefore\ exact)$$

$$R = \int_{y-constant} M^*(x,y)\,dx = \int_{y-constant} \left(\frac{y}{x^2}\right)dx = -\frac{y}{x}$$

$$K = \int N^*(0,y)\,dy = C_1$$

$$R + K = C \quad \Rightarrow \quad -\frac{y}{x} + C_1 = C \quad \Rightarrow \frac{y}{x} = C_2 \qquad \Leftarrow \textbf{Ans.}$$

Solution : (b) $(x+3y)dx+xdy=0$

$M=x+3y \quad \& \quad N=x$

$\frac{\partial M}{\partial y}=3 \quad \& \quad \frac{\partial N}{\partial x}=1 \quad (\therefore nonexact)$

$Integtation\ factor\ I=x^2$

$(IM)dx+(IN)dy=0 \quad \Rightarrow \quad (x^3+3x^2y)dx+x^3dy=0$

$\frac{\partial(IM)}{\partial y}=3x^2 \quad \& \quad \frac{\partial(IN)}{\partial x}=3x^2 \quad (\therefore exact)$

$f(x,y)=\int\limits_{y\ constant}(x^3+3x^2y)dx+k(y)$

$f(x,y)=\frac{x^4}{4}+x^3y+k(y)$

$IN=\frac{\partial f}{\partial y} \quad \Rightarrow \quad x^3=x^3+k'(y) \quad \Rightarrow \quad k'(y)=0$

$\therefore k(y)=C_1$

$\therefore \frac{x^4}{4}+x^3y+C_1=C \qquad \frac{x^4}{4}+x^3y=C_2$ $\Leftarrow$ **Ans.**

Solution : (c) $2xy^2dx+3x^2ydy=0$

$2xy^2dx+3x^2ydy=0$

$M=2xy^2 \quad \& \quad N=3x^2y$

$\frac{\partial M}{\partial y}=4xy \quad \& \quad \frac{\partial N}{\partial x}=6xy \quad (\therefore nonexact)$

$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{3x^2y}(4xy-6xy)=\frac{-2}{3x}=g(x) \qquad I=e^{\int g(x)dx}=\frac{1}{x^{\frac{2}{3}}}$

$\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{2xy^2}(4xy-6xy)=\frac{-1}{y}=h(y) \qquad I=e^{-\int h(y)dy}=y$

$M=y\,f(xy) \quad and \quad N=x\,g(xy) \qquad I=\frac{1}{xM-yN}=\frac{-1}{x^2y^2}$

$Integtation\ factor\ I=y \qquad (IM)dx+(IN)dy=0 \quad \Rightarrow \quad 2xy^3dx+3x^2y^2dy=0$

$\frac{\partial(IM)}{\partial y}=6xy^2 \quad \& \quad \frac{\partial(IN)}{\partial x}=6xy^2 \quad (\therefore exact)$

$R=\int\limits_{y-cons\tan t} IM\ dx=\int\limits_{y-cons\tan t}(2xy^3)\,dx=x^2y^3 \qquad \frac{\partial R}{\partial y}=3x^2y^2$

$K=\int\limits_{x-cons\tan t}\left(IN-\frac{\partial R}{\partial y}\right)dy=\int\limits_{x-cons\tan t}\left((3x^2y^2)-(3x^2y^2)\right)dy=\int\limits_{x-cons\tan t}(0)dy=C_1$

$R+K=C \quad \Rightarrow \quad x^2y^3+C_1=C\Rightarrow x^2y^3=C_2$ $\Leftarrow$ **Ans.**

2.4- Linear First-Order Differential Equations

A differential equation that can be written in the form:

$$\frac{dy}{dx}+P(x)y=Q(x) \tag{7}$$

Where $P(x)$ and $Q(x)$ are functions of x, is called a linear first order equation

Note: We can be written a linear first order equation in the form:

$$\frac{dx}{dy}+P(y)x=Q(y)$$

Procedure to solve a Linear First-Order Differential Equation

1. Rewrite the equation in standard form :

$$\frac{dy}{dx}+P(x)y=Q(x)$$

2. Find $\rho(x)$

$$\rho(x)=e^{\int P(x)dx}$$

3. Find $y(x)$

$$y(x)=\frac{1}{\rho(x)}\int \rho(x)Q(x)dx$$

Example 5 : Solve differential equations

(a) $\frac{dy}{dx}-\frac{1}{x}y=x^2$ (b) $x\frac{dy}{dx}+y=x$

(c) $cosh(x)\frac{dy}{dx}+sinh(x)y=e^{-x}$ (d) $(1+x)\frac{dy}{dx}+2y=x \quad , y(0)=1$

Solution : (a) $\frac{dy}{dx}-\frac{1}{x}y=x^2$

Solution : (b) $x\dfrac{dy}{dx}+y=x$

Solution : (c) $\cosh(x)\dfrac{dy}{dx}+\sinh(x)y=e^{-x}$

$$\cosh(x)\frac{dy}{dx}+\sinh(x)y=e^{-x}$$

$$\left(\frac{\cosh(x)}{\cosh(x)}\right)\frac{dy}{dx}+\left(\frac{\sinh(x)}{\cosh(x)}\right)y=\left(\frac{e^{-x}}{\cosh(x)}\right)\qquad \div\cosh(x)$$

$$\frac{dy}{dx}+\tanh(x)y=e^{-x}\operatorname{sech}(x)$$

$$P(x)=\tanh(x)\qquad \&\qquad Q(x)=e^{-x}\operatorname{sech}(x)$$

$$\rho(x)=e^{\int P(x)dx}=e^{\int\tanh(x)dx}=e^{\ln(\cosh(x))}=\cosh(x)$$

$$y(x)=\frac{1}{\rho(x)}\int\rho(x)Q(x)dx$$

$$y(x)=\frac{1}{\cosh(x)}\int\cosh(x)e^{-x}\operatorname{sech}(x)dx$$

$$y(x)=\frac{1}{\cosh(x)}\int e^{-x}dx=\frac{1}{\cosh(x)}\left(-e^{-x}+C\right)=\frac{1}{\left(\frac{e^{x}+e^{-x}}{2}\right)}\left(-e^{-x}+C\right)$$

$$y(x)=\frac{C_1e^{x}-2}{e^{2x}+1}$$ $\Leftarrow$ **Ans.**

Solution : (d) $(1+x)\frac{dy}{dx}+2y=x \quad , \; y(0)=1$

$$(1+x)\frac{dy}{dx}+2y=x$$

$$\frac{dy}{dx}+\frac{2}{1+x}y=\frac{x}{1+x}$$

$$P(x)=\frac{2}{1+x} \qquad \& \qquad Q(x)=\frac{x}{1+x}$$

$$\rho(x)=e^{\int P(x)dx}=e^{\int\frac{2}{1+x}dx}=e^{2ln(1+x)}=(1+x)^2$$

$$y(x)=\frac{1}{(1+x)^2}\int(1+x)^2\left(\frac{x}{1+x}\right)dx=\frac{1}{(1+x)^2}\int x(1+x)dx=\frac{1}{(1+x)^2}\int\left(x+x^2\right)dx$$

$$y(x)=\frac{\left(\frac{x^2}{2}+\frac{x^3}{3}\right)+C}{(1+x)^2} \quad\Rightarrow\quad 1=\frac{\left(\frac{0^2}{2}+\frac{0^3}{3}\right)+C}{(1+0)^2} \quad\Rightarrow\quad C=0$$

$$y(x)=\left(\frac{x}{1+x}\right)^2\left(\frac{1}{2}+\frac{x}{3}\right)$$ $\Leftarrow$ **Ans.**

H.W 5: Solve differential equations

(a) $x\frac{dy}{dx}+3y=\frac{sin(x)}{x^2}$

(b) $e^{2y}dx+2\left(xe^{2y}-y\right)dy=0$

(c) $\left(1+y^2\right)dx+(1+2xy)dy=0$

(d) $(x-1)^3\frac{dy}{dx}+4(x-1)^2 y=x+1$

(e) $x\frac{d^2y}{dx^2}+\frac{dy}{dx}=x^2$

(f) $x\frac{dy}{dx}+2y=1+x^2 \; , \; y(1)=1$

2.5- Bernoulli Equations

A Bernoulli differential equation has the form:

$$y' + P(x)y = Q(x)y^{n} \qquad (8)$$

Where n is a real number. The substitution

$$z = y^{1-n}$$

$$y' + P(x)y = Q(x)y^{n}$$

$$\frac{y'}{y^{n}} + P(x)\frac{1}{y^{n-1}} = Q(x)$$

$$Let \quad z = \frac{1}{y^{n-1}} \qquad \frac{dz}{dx} = (1-n)\frac{y'}{y^{n}} \qquad \therefore \frac{y'}{y^{n}} = \frac{1}{(1-n)}\frac{dz}{dx}$$

$$\frac{1}{(1-n)}\frac{dz}{dx} + P(x)z = Q(x)$$

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)$$

Procedure to solve a Bernoulli Differential Equation

1. Rewrite the equation in standard form :

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)$$

$$\frac{dz}{dx} + P^{*}(x)z = Q^{*}(x)$$

$$Where \qquad P^{*}(x) = (1-n)P(x) \quad , \quad Q^{*}(x) = (1-n)Q(x)$$

2. Find $\rho(x)$

$$\rho(x) = e^{\int P^{*}(x)dx}$$

3. Find $y(x)$

$$z(x) = \frac{1}{\rho(x)}\int \rho(x)Q^{*}(x)dx$$

Then

$$(y(x))^{(1-n)} = \frac{1}{\rho(x)}\int \rho(x)Q^{*}(x)dx$$

Example 6 : Solve differential equations

(a) $y\dfrac{dy}{dx}+y^{2}=x$ (b) $\dfrac{dy}{dx}+\dfrac{y}{x}=y^{2}$ (c) $\dfrac{dy}{dx}-\dfrac{1}{x}y=-\dfrac{\cos x}{x^{3}}y^{4}$

Solution : (a) $y\dfrac{dy}{dx}+y^{2}=x$

Solution : (b) $\dfrac{dy}{dx}+\dfrac{y}{x}=y^{2}$

Solution : (c) $\frac{dy}{dx}-\frac{1}{x}y=-\frac{\cos x}{x^3}y^4$

$$\frac{dy}{dx}-\frac{1}{x}y=-\frac{\cos x}{x^3}y^4$$

$$\frac{y'}{y^4}+\left(-\frac{1}{x}\right)\frac{1}{y^3}=-\frac{\cos x}{x^3}$$

$$n=4 \qquad Let \quad z=\frac{1}{y^3}$$

$$\frac{dz}{dx}+P^*(x)z=Q^*(x)$$

$$\frac{dz}{dx}+(1-4)\left(-\frac{1}{x}\right)z=(1-4)\left(-\frac{\cos x}{x^3}\right) \qquad \frac{dz}{dx}+\left(\frac{3}{x}\right)z=\left(\frac{3\cos x}{x^3}\right)$$

$$P^*(x)=\frac{3}{x}$$

$$Q^*(x)=\frac{3\cos x}{x^3}$$

$$\rho(x)=e^{\int P^*(x)dx}=e^{\int\left(\frac{3}{x}\right)dx}=x^3$$

$$z(x)=\frac{1}{\rho(x)}\int\rho(x)Q^*(x)dx$$

$$z(x)=\frac{1}{x^3}\int x^3\left(\frac{3\cos x}{x^3}\right)dx=\frac{1}{x^3}(3\sin x+C)$$

$$\frac{1}{y^3}=\frac{1}{x^3}(3\sin x+C)$$

$$\frac{x^3}{y^3}=3\sin x+C \qquad \Leftarrow \textbf{Ans.}$$

H.W 6: Solve differential equations

(a) $2y-3\frac{dy}{dx}=y^4e^{3x}$

(b) $xdy+(3y-x^3y^2)dx=0$

(c) $(x-2y)dy+ydx=0$

(d) $(\sin^2 x-y)dx-\tan x dy=0$

(e) $(x^2+x)dy=(x^5+3xy+3y)dx$

(f) $y''+2y'=4x$

(g) $4y(y')^2y''=(y')^4+3$

3- Second Order Linear Homogeneous Equations

The linear equation

$$\frac{d^2y}{dx^2}+2a\frac{dy}{dx}+by=0 \qquad \Rightarrow \qquad \left(D^2+2aD+b\right)y=0$$

is called ***second order linear homogenous equation***

D is called a ***linear differential operator***

The equation

$$r^2+2ar+b=0$$

is the ***characteristic equation*** of the equation

$$\frac{d^2y}{dx^2}+2a\frac{dy}{dx}+by=0$$

Solution of $\frac{d^2y}{dx^2}+2a\frac{dy}{dx}+by=0$ Roots of $r^2+2ar+b=0$	Solution
r_1, r_2 **real and unequal**	$y=C_1e^{r_1x}+C_2e^{r_2x}$
r_1, r_2 **real and equal**	$y=\left(C_1x+C_2\right)e^{r_2x}$
r_1, r_2 **complex conjugate ,** $\alpha\pm\beta i$	$y=e^{\alpha x}\left(C_1\cos\beta x+C_2\sin\beta x\right)$

Example 11 : Solve differential equation

(a) $y''+2y'=0$ (b) $y''-4y'+4y=0$ (c) $y''-2y'+4y=0$

Solution : (a) $y''+2y'=0$

$$y''+2y'=0$$
$$\left(D^2+2D\right)y=0$$
$$\left(r^2+2r\right)=0$$
$$r\left(r+2\right)=0$$
$$\therefore r_1=0 \;\&\; r_2=-2 \qquad \left(r_1 \,\&\, r_2 \;\; real\ and\ unequal\right)$$
$$y=C_1e^{0x}+C_2e^{-2x} \qquad \Rightarrow \qquad y=C_1+C_2e^{-2x}$$

$\Leftarrow$ **Ans.**

Solution : (b) $y''-4y'+4y=0$

$$y''-4y'+4y=0$$
$$\left(D^2-4D+4\right)y=0$$
$$\left(r^2-4r+4\right)=0$$
$$\left(r-2\right)^2=0$$
$$\therefore r_1=r_2=2 \qquad \left(r_1 \,\&\, r_2 \;\; real\ and\ equal\right)$$
$$y=\left(C_1x+C_2\right)e^{r_2x} \;\Rightarrow\; y=C_1+C_2e^{2x}$$

$\Leftarrow$ **Ans.**

Solution : (c) $y''-2y'+4y=0$

$$y''-2y'+4y=0$$
$$\left(D^2-2D+4\right)y=0$$
$$\left(r^2-2r+4\right)=0$$
$$r=\frac{-(-2)\pm\sqrt{(-2)^2-4\times1\times4}}{2\times1}$$
$$\therefore r_1=1\pm\sqrt{3}\,i \qquad \left(r_1\,\&\,r_2\ \ complex\ conjugate\ \right)$$
$$\alpha=1\ ,\ \beta=\sqrt{3}$$
$$y=e^{\alpha x}\left(C_1\cos\beta x+C_2\sin\beta x\right) \quad\Rightarrow\quad y=e^{x}\left(C_1\cos\sqrt{3}x+C_2\sin\sqrt{3}x\right) \qquad \Leftarrow \textbf{Ans.}$$

H.W 5 : Solve differential equation

(a) $y''-9y=0 \qquad y(\ln 2)=1\ ,\ y'(\ln 2)=-3$

(b) $4y''+12y'+9y=0 \qquad y(0)=0\ ,\ y'(0)=1$

(c) $y''-6y'+10y=0 \qquad y(0)=7\ ,\ y'(0)=1$

4- Second Order Linear Nonhomogeneous DE

If, the linear equation $F(x) \neq 0$

$$\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + by = F(x)$$

is called ***second order linear nonhomogeneous equation***

The general solution of the nonhomogeneous equation is

$$y = y_h + y_p$$

$$Where$$

$$y_h = homogeneous \ \ solution$$

$$y_p = particular\,solution$$

4.1- Method of Variation of Parameters

Steps of solution second order linear nonhomogeneous equation by variation of parameters method:

Procedure to use a Method of Variation of Parameters

1. Find $y_h = C_1\, y_1(x) + C_2\, y_2(x)$
2. Calculate

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = -y_2\, f(x)$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = +y_1\, f(x)$$

3. Find u_1 and u_2

$$u_1 = \int \frac{W_1}{W}\, dx \quad , \quad u_2 = \int \frac{W_2}{W}\, dx$$

4. Write the particular solution as $\left[y_p = u_1 y_1 + u_2 y_2\right]$
5. Write the general solution as $\left[y = y_h + y_p\right]$

Example 1 : Solve differential equation

(a) $y''-y=e^{x}$ (b) $y''-y'=e^{x}\cos x$ (c) $y''+y=\sec x\tan x$

Solution : (a) $y''-y=e^{x}$

$$y''-y=e^{x}$$

First , find y_h

$$\left(D^2-1\right)y=0$$

$$r^2-1=0 \quad\Rightarrow\quad r_1=1 \quad \& \quad r_2=-1$$

$$y=C_1e^{x}+C_2e^{-x}$$

$$\therefore u_1=e^{x} \quad , \quad u_2=e^{-x} \quad , \quad F(x)=e^{x}$$

$$\therefore u_1'=e^{x} \quad , \quad u_2'=-e^{-x}$$

$$D=\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} \Rightarrow D=\begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix}=-e^{x}e^{-x}-e^{x}e^{-x}=-1-1=-2$$

$$v_1'=\frac{-u_2F(x)}{D}\Rightarrow v_1'=\frac{-e^{-x}e^{x}}{-2}=\frac{1}{2} \quad \Rightarrow v_1=\frac{x}{2}+C_1$$

$$v_2'=\frac{u_1F(x)}{D}\Rightarrow v_2'=\frac{e^{x}e^{x}}{-2}=-\frac{1}{2}e^{2x} \quad \Rightarrow v_2=C_2-\frac{1}{4}e^{2x}$$

$$y=v_1u_1+v_2u_2$$

$$y=\left(\frac{x}{2}+C_1\right)e^{x}+\left(C_2-\frac{1}{4}e^{2x}\right)e^{-x} \qquad \left(where\ C_3=C_1-\frac{1}{4}\right)$$

$$y=\frac{1}{2}xe^{x}+C_3e^{x}+C_2e^{-x}$$ $\Leftarrow$ **Ans.**

Solution : (b) $y''-y'=e^{x}\cos x$

$$y''-y'=e^{x}\cos x$$

First , find y_h

$$\left(D^2-D\right)y=0$$

$$r^2-r=0 \quad \Rightarrow r(r-1)=0 \quad \Rightarrow r_1=0 \quad \& \quad r_2=1$$

$$y=C_1+C_2e^{x}$$

$$\therefore u_1=1 \quad , \quad u_2=e^{x} \quad , \quad F(x)=e^{x}\cos x$$

$$\therefore u_1'=0 \quad , \quad u_2'=e^{x}$$

$$D=\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} \Rightarrow D=\begin{vmatrix} 1 & e^{x} \\ 0 & e^{x} \end{vmatrix}=e^{x}-0 \qquad \Rightarrow D=e^{x}$$

$$v_1'=\frac{-u_2F(x)}{D}\Rightarrow v_1'=\frac{-e^{x}\times e^{x}\cos x}{e^{x}}=-e^{x}\cos x \qquad \Rightarrow v_1=C_1-\frac{1}{2}e^{x}\left(\cos x+\sin x\right)$$

$$v_2'=\frac{u_1F(x)}{D}\Rightarrow v_2'=\frac{1\times e^{x}\cos x}{e^{x}}=\cos x \qquad \Rightarrow v_2=C_2+\sin x$$

$$y = \left(C_1 - \frac{1}{2}e^x\left(\cos x + \sin x\right) \right) \times 1 + \left(C_2 + \sin x\right) \times e^x$$

$$y = C_1 + C_2 e^x + \frac{1}{2}e^x\left(\cos x - \sin x\right)$$ ⇐ **Ans.**

Solution : (c) $y'' + y = \sec x \tan x$

$y'' + y = \sec x \tan x$

First , find y_h

$y'' + y = 0$

$\left(D^2 + 1\right) y = 0$

$r^2 + 1 = 0 \quad \Rightarrow \quad r = \pm i \ \ (\alpha = 0, \beta = 0)$

$y = C_1 \cos x + C_2 \sin x$

$\therefore u_1 = \cos x \quad , \quad u_2 = \sin x \quad , \quad F(x) = \sec x \tan x$

$\therefore u_1' = -\sin x \quad , \quad u_2' = \cos x$

$D = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} \quad \Rightarrow \quad D = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x - \left(-\sin^2 x\right) \quad \Rightarrow \quad D = 1$

$v_1' = \dfrac{-u_2 F(x)}{D} \Rightarrow \quad v_1' = \dfrac{-\sin x \times \sec x \tan x}{1} = -\tan^2 x \quad \Rightarrow v_1 = x - \tan x + C_1$

$v_2' = \dfrac{u_1 F(x)}{D} \quad \Rightarrow \quad v_2' = \dfrac{\cos x \times \sec x \tan x}{1} = \tan x \quad \Rightarrow v_2 = \ln\left|\sec x\right| + C_2$

$y = \left(x - \tan x + C_1\right) \times \cos x + \left(\ln\left|\sec x\right| + C_2\right) \times \sin x \qquad \left(C_3 = C_2 - 1\right)$

$y = x \cos x + \sin x \ln\left|\sec x\right| + C_1 \cos x + C_3 \sin x$ ⇐ **Ans.**

H.W 6 : Solve differential equation

(a) $y'' + y = \csc x$

(b) $y'' + y = \cot x$

(c) $y'' - y' = e^x + e^{-x}$

(d) $y'' - 4y' - 5y = e^x + 4$

(e) $y'' + y = \sec^2 x$

4.2- Method of Undetermined Coefficients

Procedure to use a Method of Undetermined Coefficients

2. Find y_h
3. Find y_p
4. Write the general solution as $\left[y = y_h + y_p\right]$

$F(x)$	Condition	y_p
e^{kx}	$k \neq r_1$ and $k \neq r_2$	Ae^{kx}
	$k = r_1$ or $k = r_2$	Axe^{kx}
	$k = r_1$ and $k = r_2$	Ax^2e^{kx}
$\sin kx, \cos kx$	$k \neq r_1$ and $k \neq r_2$	$B\cos kx + C\sin kx$
	$k = r_1$ and $k = r_2$	$Bx\cos kx + Cx\sin kx$
$ax^2 + bx + c$	$0 \neq r_1$ and $0 \neq r_2$	$Dx^2 + Ex + F$
	$0 = r_1$ or $0 = r_2$	$Dx^3 + Ex^2 + Fx$
	$0 = r_1$ and $0 = r_2$	$Dx^4 + Ex^3 + Fx^2$

Example 7 : Solve differential equation

(a) $y'' - y = e^x + x^2$ (b) $y'' - y' - 6y = e^{-x} - 7\cos x$

Solution : (a) $y'' - y = e^x + x^2$

$$y'' - y = e^x + x^2$$

$$First, find\ y_h$$

$$y'' - y = 0$$

$$\left(D^2 - 1\right)y = 0$$

$$r^2 - 1 = 0 \quad \Rightarrow \quad r_1 = 1 \quad \& \quad r_2 = -1$$

$$y = C_1 e^x + C_2 e^{-x}$$

$$Second, find\ y_p$$

$$\because F(x)_1 = e^x \ , \quad k = 1 = r_1 \quad \Rightarrow \quad \therefore \left(y_p\right)_1 = Axe^x$$

$$\because F(x)_2 = x^2 \ , \quad r_1 \,\&\, r_2 \neq 0 \quad \Rightarrow \quad \therefore \left(y_p\right)_2 = Bx^2 + Cx + D$$

$$y_p = (y_p)_1 + (y_p)_2$$
$$y_p = Axe^x + Bx^2 + Cx + D$$
$$y_p' = A\left(xe^x + e^x\right) + 2Bx + C$$
$$y_p'' = A\left(xe^x + 2e^x\right) + 2B = Axe^x + 2Ae^x + 2B$$
$$y_p'' - y_p = e^x + x^2$$
$$\left(Axe^x + 2Ae^x + 2B\right) - \left(Axe^x + Bx^2 + Cx + D\right) = e^x + x^2$$
$$2Ae^x - Bx^2 - Cx + 2B - D = e^x + x^2$$
$$\therefore \quad 2Ae^x = e^x \quad \Rightarrow \quad A = \frac{1}{2}$$
$$\therefore \quad -Bx^2 = x^2 \quad \Rightarrow \quad B = -1$$
$$\therefore \quad -Cx = 0x \quad \Rightarrow \quad C = 0$$
$$\therefore \quad 2B - D = 0 \quad \Rightarrow \quad D = 2B = -2$$
$$\therefore \quad y_p = \frac{1}{2}xe^x - x^2 - 2$$
$$\therefore \quad y = y_h + y_p \quad \Rightarrow \quad y = C_1 e^x + C_2 e^{-x} + \frac{1}{2}xe^x - x^2 - 2$$ $\Leftarrow$ **Ans.**

Solution : (b) $y'' - y' - 6y = e^{-x} - 7\cos x$

$$y'' - y' - 6y = e^{-x} - 7\cos x$$
First , find y_h
$$\left(D^2 - D - 6\right)y = 0$$
$$r^2 - r - 6 = 0 \quad \Rightarrow \quad (r-3)(r+2) = 0 \quad \Rightarrow \quad r_1 = 3 \quad \& \quad r_2 = -2$$
$$y = C_1 e^{3x} + C_2 e^{-2x}$$
Second , find y_p
$$\because F(x)_1 = e^{-x} \quad , \quad k = -1 \quad (r_1 \,\&\, r_2 \neq k) \quad \Rightarrow \quad \therefore (y_p)_1 = Ae^{-x}$$
$$\because F(x)_2 = \cos x \quad , \quad k = 1 \quad (r_1 \,\&\, r_2 \neq k) \quad \Rightarrow \quad \therefore (y_p)_2 = B\cos x + C\sin x$$
$$y_p = (y_p)_1 + (y_p)_2 \quad \Rightarrow \quad y_p = Ae^{-x} + B\cos x + C\sin x$$
$$y_p' = -Ae^{-x} - B\sin x + C\cos x$$
$$y_p'' = Ae^{-x} - B\cos x - C\sin x$$
$$\because y_p'' - y_p' - 6y_p = e^{-x} - 7\cos x$$
$$\left(Ae^{-x} - B\cos x - C\sin x\right) - \left(-Ae^{-x} - B\sin x + C\cos x\right) - 6\left(Ae^{-x} + B\cos x + C\sin x\right) = e^{-x} - 7\cos x$$
$$(1+1-6)Ae^{-x} + (-B - C - 6B)\cos x + (-C + B - 6C)\sin x = e^{-x} - 7\cos x$$

$$\therefore\ -4Ae^{-x} = e^{-x} \qquad \Rightarrow \qquad A = -\frac{1}{4}$$

$$\therefore\ -(7B + C)\cos x = -7\cos x \qquad \Rightarrow \qquad 7B + C = 7 \quad .....(1)$$

$$\therefore\ (B - 7C)\sin x = 0\sin x \qquad \Rightarrow \qquad B - 7C = 0 \quad .....(2)$$

$$\therefore\ B = \frac{49}{50} \quad , \quad C = \frac{7}{50}$$

$$\therefore\ y_p = -\frac{1}{4}e^{-x} + \frac{49}{50}\cos x + \frac{7}{50}\sin x$$

$$\therefore\ y = y_h + y_p \qquad \Rightarrow \qquad y = C_1 e^{3x} + C_2 e^{-2x} - \frac{1}{4}e^{-x} + \frac{49}{50}\cos x + \frac{7}{50}\sin x \qquad \Leftarrow \textbf{Ans.}$$

Differential Equation	y_p
$y''+2ay'+by=c$	$\frac{c}{b}$
$y''+2ay'=c$	$\frac{c}{2a}x$
$y''+2ay'+by=cx+d$	$\frac{c}{b}x+\frac{bd-2ac}{b^2}$
$y''+2ay'=cx+d$	$\frac{c}{4a}x^2+\frac{2ad-c}{4a^2}x$
$y''=cx+d$	$\frac{c}{6}x^3+\frac{d}{2}x^2$
$y''+2ay'+by=ce^{kx}$	$\frac{c}{k^2+2ak+b}e^{kx}$
$y''+2ay'+by=ce^{kx} \quad k=r_1 \text{ or } r_2$	$\frac{c}{2a+2k}xe^{kx}$
$y''+2ay'+by=ce^{kx} \quad k=r_1=r_2$	$\frac{c}{2}x^2e^{kx}$
$y''+2ay'+by=c\cos kx \quad k\neq r_1\neq r_2$	$\left(\frac{c}{(b-k^2)^2+(2ak)^2}\right)\left((b-k^2)\cos kx+(2ak)\sin kx\right)$
$y''+2ay'+by=c\cos kx \quad k=r_1=r_2$	$-\frac{c}{2k^2}\sin kx$
$y''+k^2y=c\cos kx$	$\frac{c}{2k}x\sin kx$
$y''+2ay'+by=c\sin kx \quad k\neq r_1\neq r_2$	$\left(\frac{c}{(b-k^2)^2+(2ak)^2}\right)\left((-2ak)\cos kx+(b-k^2)\sin kx\right)$
$y''+2ay'+by=c\sin kx \quad k=r_1=r_2$	$\frac{c}{2k^2}\cos kx$
$y''+k^2y=c\sin kx$	$-\frac{c}{2k}x\cos kx$
$y''+2ay'+by=cx^2+dx+e$	$\frac{c}{b}x^2+\frac{bd-4ac}{b^2}x+\frac{8a^2c-2abd+b^2e-2bc}{b^3}$
$y''+2ay'=cx^2+dx+e$	$\frac{c}{6a}x^3+\frac{ad-c}{4a^2}x^2+\frac{2a^2e-ad+c}{4a^3}x$
$y''=cx^2+dx+e$	$\frac{c}{12}x^4+\frac{d}{6}x^3+\frac{e}{2}x^2$