CHAPTER 1

PRELIMINARIES

1.1 Real Numbers and the Real Line

Calculus is based on the real number system. Real numbers are numbers that can be expressed as decimals.
We distinguish three special subsets of real numbers:

1. The **natural numbers**, namely 1, 2, 3, 4,…
2. The **integers**, namely 0, ±1, ±2, ±3,…
3. The **rational numbers**, which are ratios of integers. These numbers can be expressed in the form of a function *m*/*n,* where *m* and *n* are integers and *n*≠0. Examples are:

$$\frac{1}{2}, -\frac{5}{3}=\frac{-5}{3}=\frac{5}{-3}, \frac{200}{13}, 67=\frac{67}{1}$$

(Recall that division by is always ruled out, so expressions like $\frac{3}{0}$ and $\frac{0}{0}$ are undefined.)

The real numbers can be represented geometrically as points on a number line called the **real line,** as in Figure 1.1.

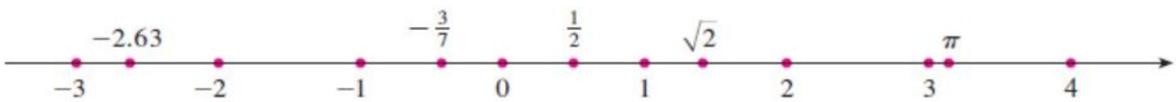


Figure 1.1

1.1.1 Intervals

Certain sets (or a subset) of real numbers, called intervals, occur frequently in calculus and correspond geometrically to line segments. For example, if $a < b$, the **open interval** from to consists of all numbers between a and b is denoted by the symbol (a, b). Using set-builder notation, we can write:

$$(a, b) = \{x|\ a < x < b\}$$

(which is read "(*a, b*) is the set of *x* such that *x* is an integer and $a < x < b$.)

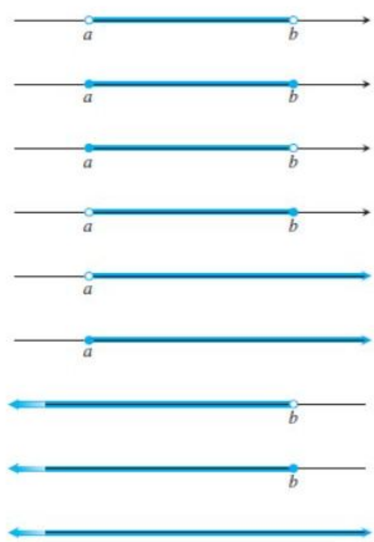
Notice that the endpoints of the interval -namely, *a* and *b*- are excluded. This is indicated by the round brackets and by the open dots in Table 1.1. The **closed interval** from *a* to *b* is the set

$$[a, b] = \{x|\ a \le x \le b\}$$

Here the endpoints of the interval are included. This is indicated by the square brackets [ ] and by the solid dots in table 1.1. It is also possible to include only one endpoint in an interval, as shown in Table 1.1.

Table 1.1

	Notation	Set description	Type	Picture
Finite:	(a, b)	$\{x \mid a < x < b\}$	Open	
	$[a, b]$	$\{x \mid a \le x \le b\}$	Closed	
	$[a, b)$	$\{x \mid a \le x < b\}$	Half-open	
	$(a, b]$	$\{x \mid a < x \le b\}$	Half-open	
Infinite:	(a, ∞)	$\{x \mid x > a\}$	Open	
	$[a, \infty)$	$\{x \mid x \ge a\}$	Closed	
	$(-\infty, b)$	$\{x \mid x < b\}$	Open	
	$(-\infty, b]$	$\{x \mid x \le b\}$	Closed	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	



1.1.2 Inequalities

The process of finding the interval or intervals of numbers that satisfy an inequality in *x* is called **solving** the inequality.

The following useful rules can be derived from them, where the symbol => means "implies."

Rules for Inequalities

If a, b, and c are real numbers, then:

1. $a < b \Rightarrow a + c < b + c$
2. $a < b \Rightarrow a - c < b - c$
3. $a < b$ and $c > 0 \Rightarrow ac < bc$
4. $a < b$ and $c < 0 \Rightarrow bc < ac$
   Special case: $a < b \Rightarrow -b < -a$
5. $a > 0 \Rightarrow \frac{1}{a} > 0$
6. If a and b are both positive or both negative, then $a < b \Rightarrow \frac{1}{b} < \frac{1}{a}$

Example 1: Solve the following inequalities and show their solution sets on the real line.

(a) $2x - 1 < x + 3$ **(b)** $-\frac{x}{3} < 2x + 1$ **(c)** $\frac{6}{x-1} \geq 5$

Solution:

(a)

$$2x - 1 < x + 3$$

$$2x < x + 4 \quad \text{Add 1 to both sides.}$$

$$x < 4 \quad \text{Subtract } x \text{ from both sides.}$$

The solution set is the open interval $(-\infty, 4)$ (Figure 1.1a).

(b)

$$-\frac{x}{3} < 2x + 1$$

$$-x < 6x + 3 \quad \text{Multiply both sides by 3.}$$

$$0 < 7x + 3 \quad \text{Add } x \text{ to both sides.}$$

$$-3 < 7x \quad \text{Subtract 3 from both sides.}$$

$$-\frac{3}{7} < x \quad \text{Divide by 7.}$$

The solution set is the open interval $(-3/7, \infty)$ (Figure 1.1b).

The inequality $6/(x - 1) \geq 5$ can hold only if $x > 1$ because otherwise $6/(x - 1)$ is undefined or negative. Therefore, $(x - 1)$ is positive and the inequality will be preserved if we multiply both sides by $(x - 1)$ and we have

$$\frac{6}{x-1} \geq 5$$

$$6 \geq 5x - 5 \qquad \text{Multiply both sides by } (x-1).$$

$$11 \geq 5x \qquad \text{Add 5 to both sides.}$$

$$\frac{11}{5} \geq x. \qquad \text{Or } x \leq \frac{11}{5}.$$

The solution set is the half-open interval (1, 11/5] (Figure 1.1c)

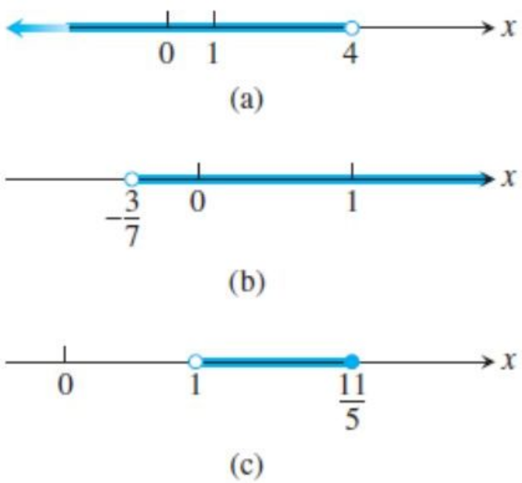


Figure 1.2

1.1.3 Absolute Value

The **absolute value** of a number x, denoted by $|x|$, is the distance from x to 0 on the real number line. Distances are always positive or 0, so we have

$$|x| \geq 0 \quad \text{for every number } x$$

Or it can be defined by the formula:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Example 2:

$|3| = 3$, $|0| = 0$, $|-5| = -(-5) = 5$, $|-|a|| = |a|$

Geometrically, the absolute value of x is the distance from x to 0 on the real number line. Since distances are always positive or 0, we see that $|x| \geq 0$ for every real number x, and $|x| = 0$ if and only if $x = 0$. Also,

$$|x - y| = \text{the distance between } x \text{ and } y$$

on the real line (Figure 1.2).

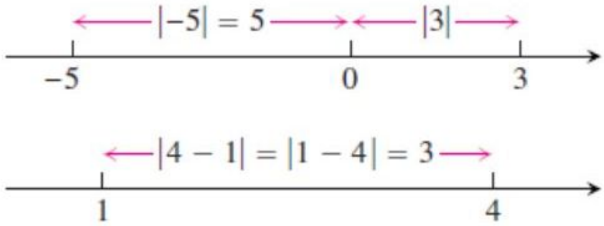


Figure 1.3

The absolute value has the following properties:

Absolute Value Properties

1. $|-a| = |a|$	A number and its additive inverse or negative have the same absolute value.
2. $|ab| = |a||b|$	The absolute value of a product is the product of the absolute values.
3. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$	The absolute value of a quotient is the quotient of the absolute values.
4. $|a + b| \leq |a| + |b|$	The **triangle inequality**. The absolute value of the sum of two numbers is less than or equal to the sum of their absolute values.

Example 3:

$$|-3+5| = |2| = 2 < |-3| + |5| = 8$$
$$|3+5| = |8| = |3| + |5|$$
$$|-3-5| = |-8| = 8 = |-3| + |-5|$$

The following statements are all consequences of the definition of absolute value and are often helpful when solving equations or inequalities involving absolute values:

Absolute Values and Intervals

If a is any positive number, then

5. $|x| = a$ if and only if $x = \pm a$
6. $|x| < a$ if and only if $-a < x < a$
7. $|x| > a$ if and only if $x > a$ or $x < -a$
8. $|x| \le a$ if and only if $-a \le x \le a$
9. $|x| \ge a$ if and only if $x \ge a$ or $x \le -a$

The inequality $|x| < a$ *says* that the distance from x to 0 is less than the positive number a. This means that x must lie between $-a$ and a, as we can see from Figure 1.4.

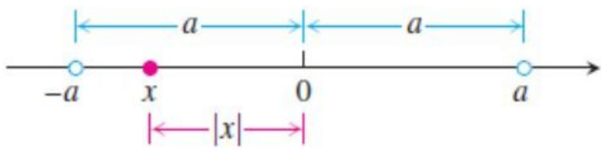


Figure 1.4

Example 4: Solve the equation $|2x - 3| = 7$

Solution:

By Property 5, $2x - 3 = \pm 7$, so there are two possibilities:

$2x - 3 = 7$	$2x - 3 = -7$	Equivalent equations without absolute values
$2x = 10$	$2x = -4$	Solve as usual.
$x = 5$	$x = -2$	

The solutions of $|2x - 3| = 7$ are $x = 5$ and $x = -2$

Example 5: Solve the inequality $\left|5-\frac{2}{x}\right|<1$

Solution We have

$$\left|5-\frac{2}{x}\right|<1 \Leftrightarrow -1<5-\frac{2}{x}<1 \qquad \text{Property 6}$$

$$\Leftrightarrow -6<-\frac{2}{x}<-4 \qquad \text{Subtract 5.}$$

$$\Leftrightarrow 3>\frac{1}{x}>2 \qquad \text{Multiply by } -\frac{1}{2}.$$

$$\Leftrightarrow \frac{1}{3}<x<\frac{1}{2}. \qquad \text{Take reciprocals.}$$

(The symbol <=> is often used by mathematicians to denote the "if and only if" logical relationship. It also means "implies and is implied by.")
The original inequality holds if and only if $(1/3) < x < (1/2)$. The solution set is the open interval (1/3, 1/2).

1.2 Lines, Circles, and Parabolas

1.2.1 Coordinate Geometry and Lines

The points in a plane can be identified with ordered pairs of real numbers. We start by drawing two perpendicular coordinate lines that intersect at the origin *O* on each line. Usually one line is horizontal with positive direction to the right and is called the ***x*-axis**; the other line is vertical with positive direction upward and is called the ***y*-axis**.
Any point ***P*** in the plane can be located by a unique ordered pair of numbers as follows:
Draw lines through ***P*** perpendicular to the *x*- and *y*-axes. These lines intersect the axes in points with coordinates and as shown in Figure 1.5. Then the point *P* is assigned the ordered pair (*a, b*). The first number *a* is called the ***x*-coordinate** (or **abscissa**) of *P*; the second number *b* is called the ***y*-coordinate** (or **ordinate**) of *P*. We say that *P* is the point with coordinates (*a, b*), and we denote the point by the symbol *P* (*a, b*). Several points are labeled with their coordinates in Figure 1.6.

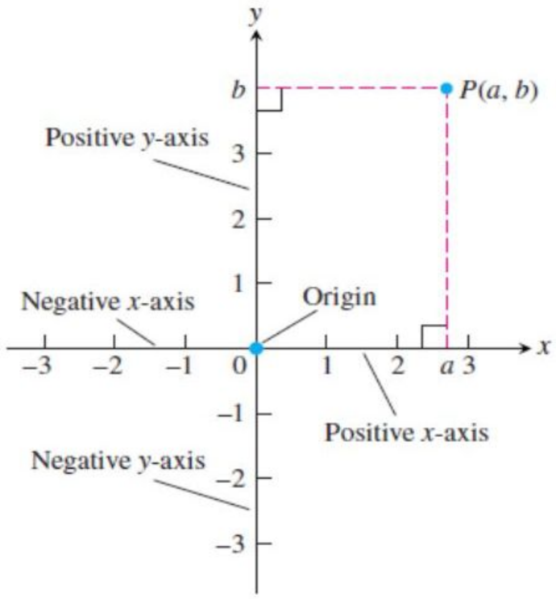


Figure 1.5

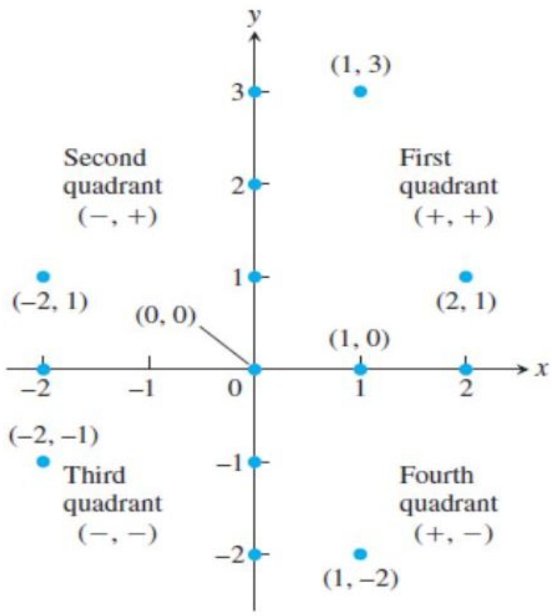


Figure 1.6

This coordinate system is called the **rectangular coordinate system** or the **Cartesian coordinate system**.

The plane supplied with this coordinate system is called the **coordinate plane** or the **Cartesian plane**.

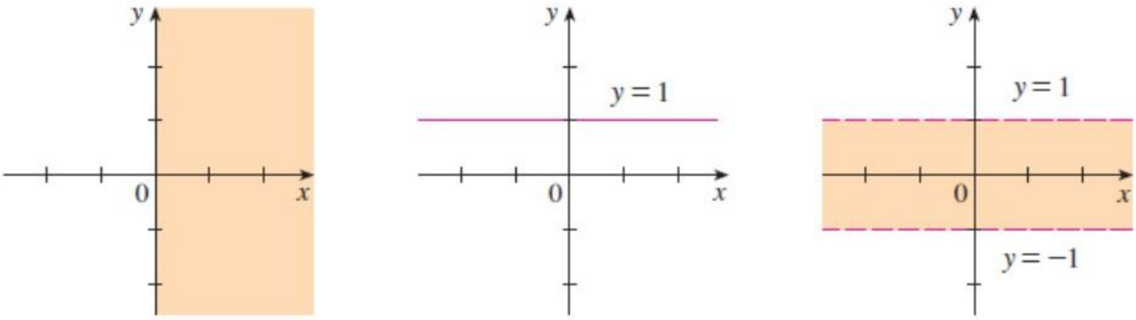
The *x*- and *y*-axes are called the coordinate axes and divide the Cartesian plane into four quadrants: First quadrant, Second quadrant, Third quadrant and Fourth quadrant as shown in Figure 1.6. Notice that the First quadrant consists of those points whose *x*- and *y*-coordinates are both positive.

Example 6: Describe and sketch the regions given by the following sets:

(a) $\{(x, y) \,/\, x \geq 0\}$ (b) $\{(x, y) \,/\, y = 1\}$ (c) $\{(x, y) \,/\, |y| < 1\}$

Solution:

(a) The points whose *x*-coordinates are 0 or positive lie on the *y*-axis or to the right of it as indicated by the shaded region in Figure 1.7 (a).



(a) $x \geq 0$ (b) $y = 1$ (c) $|y| < 1$

Figure 1.7

(b) The set of all points with *y*-coordinate 1 is a horizontal line one unit above the *x*-axis [see Figure 1.7(b)].

(c) $|y| < 1$ if and only if $-1 < y < 1$

The given region consists of those points in the plane whose *y*-coordinates lie between -1 and 1. Thus the region consists of all points that lie between (but not on) the horizontal lines $y = 1$ and $y = -1$. [These lines are shown as dashed lines in Figure 1.7(c) to indicate that the points on these lines don't lie in the set.]

1.2.2 Increments and Straight Lines

When a particle moves from one point in the plane to another, the net changes in its coordinates are called *increments.* They are calculated by subtracting the coordinates of the starting point from the coordinates of the ending point. If x changes from x_1 to x_2 the **increment** in x is:

$$\Delta x = x_2 - x_1$$

Example 7: In going from the point $A(4, -3)$ to the point $B(2, 5)$ the increments in the *x*- and *y*-coordinates are

$$\Delta x = 2 - 4 = -2, \qquad \Delta y = 5 - (-3) = 8$$

From $C(5, 6)$ to $D(5, 1)$ the coordinate increments are

$$\Delta x = 5 - 5 = 0, \qquad \Delta y = 1 - 6 = -5$$

See Figure 1.8

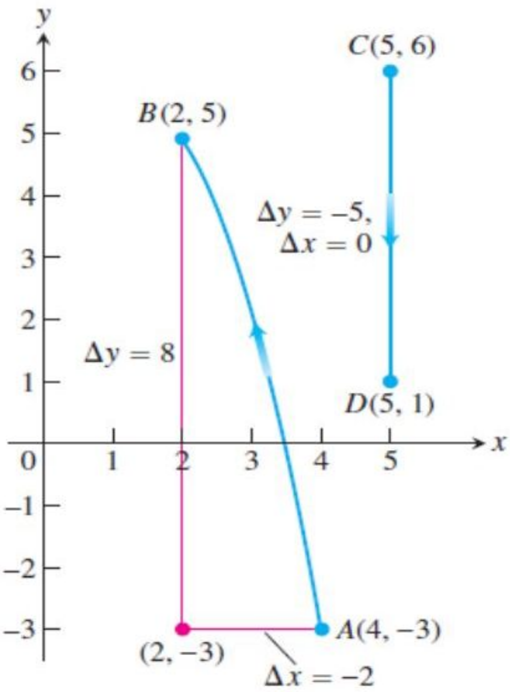


Figure 1.8

1.2.3 Slope of straight line

Slope is a measure of the steepness of the line.

Given two points $P_1\ (x_1,\ y_1)$ and $P_2\ (x_2,\ y_2)$ in the plane, we call the increments $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ the **run** and the **rise**, respectively, between P_1 and P_2. Two such points always determine a unique straight line (usually called simply a line) passing through them both. We call the line P_1 P_2.

Any nonvertical line in the plane has the property that the ratio

$$m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y2 - y1}{x2 - x1}$$

has the same value for every choice of the two points $P_1\ (x_1,\ y_1)$ and $P_2\ (x_2,\ y_2)$ on the line (Figure 1.9). This is because the ratios of corresponding sides for similar triangles are equal.

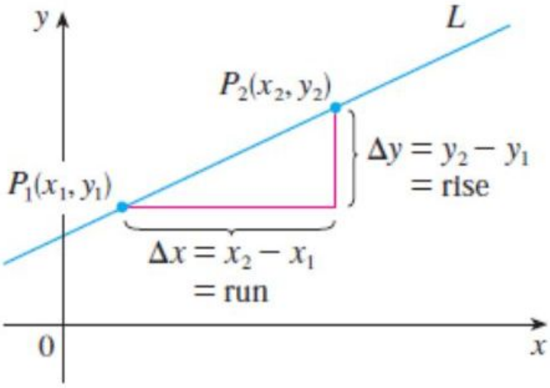


Figure 1.9

DEFINITION The **slope** of a nonvertical line that passes through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a vertical line is not defined.

Figure 1.10 shows several lines labeled with their slopes. Notice that lines with positive slope slant **upward to the right**, whereas lines with negative slope slant **downward to the right**. Notice also that the **horizontal line has slope 0** because $\Delta y = 0$ and the slop of the **vertical line is undefined** because $\Delta x = 0$.

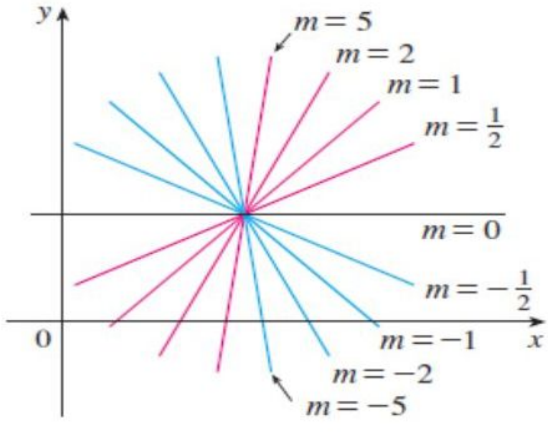


Figure 1.10

Example 8: find the slop of the nonvertical straight line L1 passes through the points $P_1(0, 5)$ and $P_2(4, 2)$ and L2 passes $P_3(0, -2)$ and $P_4(3, 6)$.

Solution:

Line *L1*:

The slope of *L1* is $m = \frac{\Delta y}{\Delta x} = \frac{y2-y1}{x2-x1}$

$$= \frac{6-(-2)}{3-0} = \frac{8}{3}$$

That is, *y* increases 8 units every time *x* increases 3 units.

Line *L2*:

The slope of *L2* is $m = \frac{\Delta y}{\Delta x} = \frac{y2-y1}{x2-x1}$

$$m = \frac{\Delta y}{\Delta x} = \frac{2-5}{4-0} = \frac{-3}{4}$$

That is, *y* decreases 3 units every time *x* increases 4 units.

Lines *L1* and *L2* explained in Figure 1.11

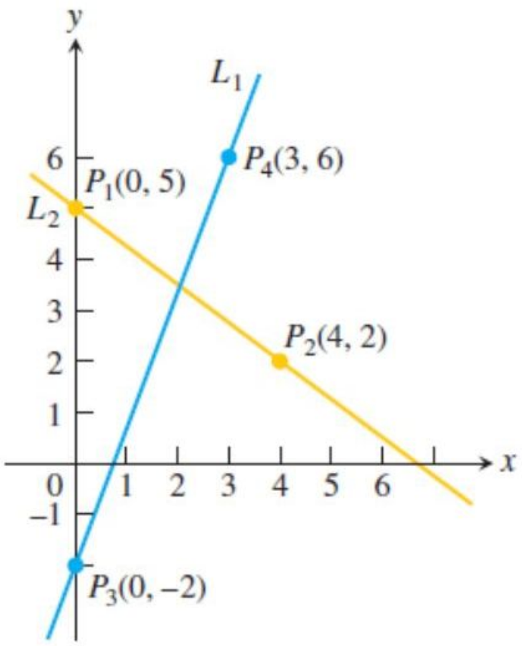


Figure 1.11

1.2.4 Equation of straight line

(a) Point-Slope Form of the Equation of a Line

Now let's find an equation of the line that passes through a given point $P_1\ (x_1, y_1)$ and has slope m. A point $P(x, y)$ with $x \neq x_1$ lies on this line if and only if the slope of the line through P_1 and P is equal to m; that is:

$$\frac{y-y1}{x-x1} = \mathrm{m}$$

This equation can be rewritten in the form:

$$y - y_1 = m\,(x - x_1)$$

and we observe that this equation is also satisfied when $x = x_1$ and $y = y_1$. Therefore it is an equation of the given line.

> The equation
>
> $$y = y_1 + m(x - x_1)$$
>
> is the **point-slope equation** of the line that passes through the point (x_1, y_1) and has slope m.

Example 9: Find an equation of the line through (1, -7) with slope – 1/2.
Solution:
Using Point-slope form of the equation of a line with $m = -1/2$, $x_1 = 1$ and $y_1 = -7$, we obtain an equation of the line as:

$$y + 7 = -1/2\ (x - 1)$$

which we can rewrite as:

$$2y + 14 = -x + 1 \quad \text{or} \quad x + 2y + 13 = 0$$

Example 10: Write an equation for the line through the point (2, 3) with slope -3/2
Solution:
We substitute $x_1 = 2$, $y_1 = 3$ and $m = -3/2$ into the point-slope equation and obtain

$$y = 3 - 3/2\,(x - 2), \quad \text{or} \quad y = -\,3/2\,(x) + 6$$

When $x = 0$, y = 6 so the line intersects the y-axis at y = 6.

(b) A Line Through Two Points

Example 11: Find an equation of the line through the points (-1, 2) and (3, - 4).
Solution:
By Definition the slope of the line:

$$m = \frac{-4 - 2}{3 - (-1)} = -\frac{3}{2}$$

Using the point-slope form with $x_1 = -1$ and $y_1 = 2$, we obtain:

$$y - 2 = -\,3/2\,(x + 1)$$

or

$$3x + 2y = 1$$

Example 12: Write an equation for the line through (-2, -1) and (3, 4).
Solution: The line's slope is

$$m = \frac{-1 - 4}{-2 - 3} = \frac{-5}{-5} = 1.$$

We can use this slope with either of the two given points in the point-slope equation:

With $(x_1, y_1) = (-2, -1)$	With $(x_1, y_1) = (3, 4)$
$y = -1 + 1 \cdot (x - (-2))$	$y = 4 + 1 \cdot (x - 3)$
$y = -1 + x + 2$	$y = 4 + x - 3$
$y = x + 1$	$y = x + 1$

Same result

Either way, $y = x + 1$ is an equation for the line (Figure 1.12)

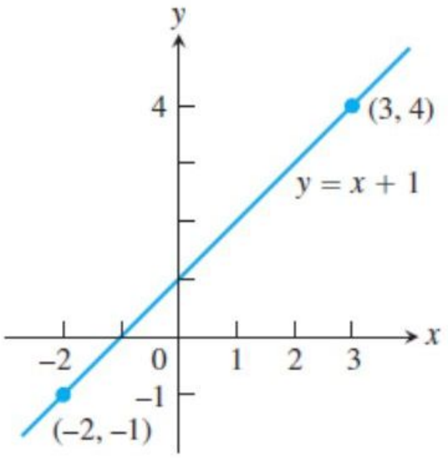


Figure 1.12

(c) Slope-Intercept Form of The Equation of a Line

Suppose a nonvertical line has slope m and y-intercept b. (See Figure 1.13).

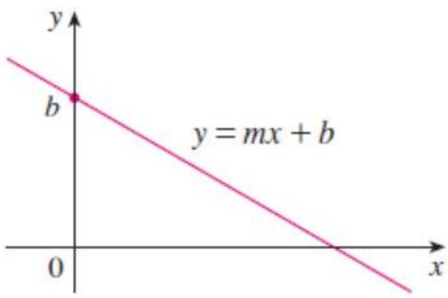


Figure 1.13

This means it intersects the y-axis at the point $(0, b)$, so the point-slope form of the equation of the line, with $x_1 = 0$ and $y_1 = 0$, becomes:

$$y - b = m\,(x - 0)$$

This simplifies as follows:

> The equation
>
> $$y = mx + b$$
>
> is called the **slope-intercept equation** of the line with slope m and y-intercept b.

Example : Find the intercepts of the axis of the equation $y = x^2 - 1$
Solution: For x-intercept, let y = 0 $\rightarrow$ $x^2 - 1 = 0$ $\rightarrow$ $x^2 = 1$ $\rightarrow$ x=±1
For y-intercept, let x = 0 $\rightarrow$ $y = 0 - 1$ $\rightarrow$ $y = -1$

In particular, if a line is horizontal, its slope is $m = 0$, so its equation is $y = b$, where b is the y-intercept (see Figure 1.14). A vertical line does not have a slope, but we can write its equation as $x = a$, where a is the x-intercept, because the x-coordinate of every point on the line is a.

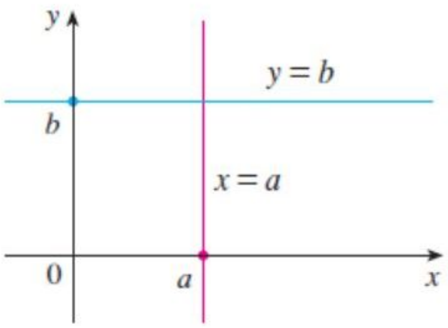


Figure 1.14

Example 13: Write the standard equations for the vertical and horizontal lines through point (2, 3).
Solution:

$(a, b) = (2, 3)$
x-intercept $= a = 2$
y-intercept $= b = 3$

- Horizontal line equation:

$y = m\,x + b$
$\;\;= 0\,x + 3$
$y = 3$

- Vertical line equation:

$x = 2$
two lines was shown in Figure 1.15

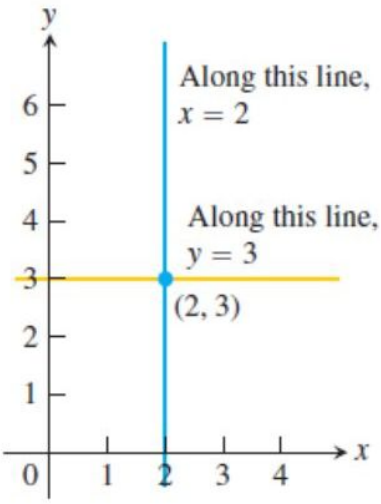


Figure 1.15

(d) The General Equation of a Line

A **linear equation** or the **general equation** of a line can be written in the form:

$$Ax + By + C = 0$$

where, A, B, and C are constants and A and B are not both 0.

We can show that it is the equation of a line:

- If $B = 0$, the equation becomes $Ax + C = 0$ or $x = - C/A$, which represents a vertical line with x-intercept $- C/A$.
- If $B \neq 0$, the equation can be rewritten by solving for y:

$$y = -\frac{A}{B}x - \frac{C}{B}$$

We recognize this as being the **slope-intercept** form of the equation of a line ($m= - A/B$, $b= - C/B$).

Example 14: Find the slope and y-intercept of the line $8x + 5y = 20$
Solution: Solve the equation for y to put it in slope-intercept form:

$$8x + 5y = 20$$
$$5y = -8x + 20$$
$$y = -\frac{8}{5}x + 4.$$

The slope is m = - 8/5. The y-intercept is b = 4

(e) Parallel and Perpendicular Lines

Slopes can be used to show that lines are parallel or perpendicular. The following facts are proved:

PARALLEL AND PERPENDICULAR LINES

1. Two nonvertical lines are parallel if and only if they have the same slope.
2. Two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$; that is, their slopes are negative reciprocals:

$$m_2 = -\frac{1}{m_1}$$

Example 15: Find an equation of the line through the point (5, 2) that is parallel to the line $4x + 6y + 5 = 0$.
Solution: The given line can be written in the form

$$y = -\frac{2}{3}x - \frac{5}{6}$$

which is in slope-intercept form with m = - 2/3. Parallel lines have the same slope, so the required line has slope – 2/3 and its equation in point-slope form is

$$y - 2 = - 2/3\ (x - 6)$$

We can write this equation as $2x + 3y = 16$.

Example 16: Show that the lines $2x + 3y = 1$ and $6x - 4y - 1 = 0$ are perpendicular.
Solution: The equations can be written as

$$y = -\tfrac{2}{3}x + \tfrac{1}{3} \qquad \text{and} \qquad y = \tfrac{3}{2}x - \tfrac{1}{4}$$

from which we see that the slopes are

$$m_1 = -\tfrac{2}{3} \qquad \text{and} \qquad m_2 = \tfrac{3}{2}$$

Since $m_1 m_2 = -1$, the lines are perpendicular.

1.2.5 Distance and Circles in the Plane

To find the distance $|P_1P_2|$ between any two points $P_1\ (x_1,\ y_1)$ and $P_2\ (x_2,\ y_2)$, we note that triangle $P_1P_2P_3$ in Figure 1.16 is a right triangle, and so by the Pythagorean Theorem we have:

$$|P1P2| = \sqrt{|P1P3|^2 + |P2P3|^2} = \sqrt{|x2 - x1|^2 + |y2 - y1|^2}$$

$$= \sqrt{(x2 - x1)^2 + (y2 - y1)^2}$$

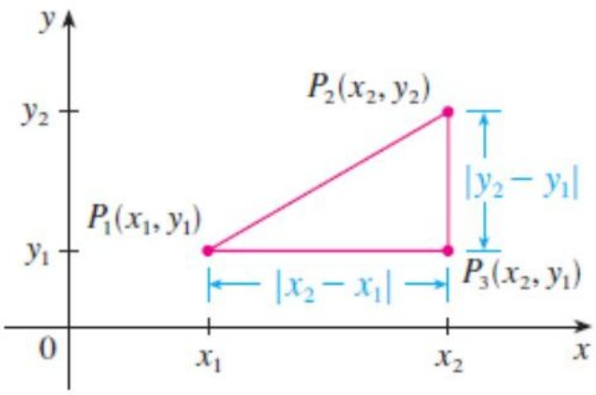


Figure 1.16

Distance Formula for Points in the Plane

The distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example 17: The distance between (1, -2) and (5, 3) is

$$\sqrt{(5-1)^2 + [3-(-2)]^2} = \sqrt{4^2 + 5^2} = \sqrt{41}$$

Second-Degree Equations

In the proceeding sections we saw that a first-degree, or linear, equation $Ax + By + C = 0$ represents a line. In this section we discuss second-degree equations such as

$$x^2 + y^2 = 1 \qquad y = x^2 + 1 \qquad \frac{x^2}{9} + \frac{y^2}{4} = 1 \qquad x^2 - y^2 = 1$$

which represent a circle, a parabola, an ellipse, and a hyperbola, respectively.

(a) Circles

To find an equation of the circle with radius r and center (h, k), by definition, the circle is the set of all points $P(x, y)$ whose distance from the center $C(h, k)$is r. (See Figure 1.17).

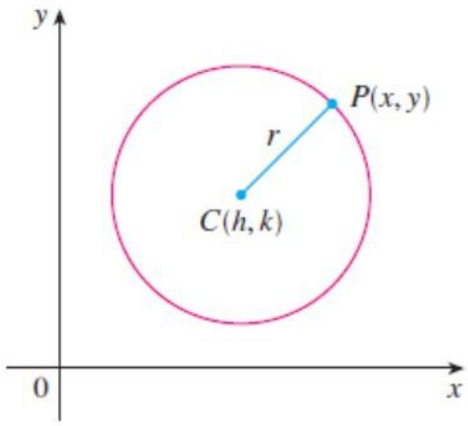


Figure 1.17

Thus P is on the circle if and only if $|PC| = r$. From the distance formula, we have:

$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

or equivalently, squaring both sides, we get

$$(x - h)^2 + (y - k)^2 = r^2$$

☐ **EQUATION OF A CIRCLE** An equation of the circle with center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2$$

In particular, if the center is the origin (0, 0), the equation is

$$x^2 + y^2 = r^2$$

Example 18:

(a) The standard equation for the circle of radius 2 centered at (3, 4) is:

$$(x - 3)^2 + (y - 4)^2 = 2^2 = 4$$

(b) The circle

$$(x - 1)^2 + (y + 5)^2 = 3$$

Has $h = 1$, $k = -5$ and $r = \sqrt{3}$. The center is the point $(h, k) = (1, -5)$ and the radius is $r = \sqrt{3}$.

Example 19: Find the center and radius of the circle

$$x^2 + y^2 + 4x - 6y - 3 = 0.$$

Solution: We convert the equation to standard form by completing the squares in x and y:

$$x^2 + y^2 + 4x - 6y - 3 = 0$$

Start with the given equation.

$$(x^2 + 4x \quad) + (y^2 - 6y \quad) = 3$$

Gather terms. Move the constant to the right-hand side.

$$\left(x^2 + 4x + \left(\frac{4}{2}\right)^2\right) + \left(y^2 - 6y + \left(\frac{-6}{2}\right)^2\right) = 3 + \left(\frac{4}{2}\right)^2 + \left(\frac{-6}{2}\right)^2$$

Add the square of half the coefficient of x to each side of the equation. Do the same for y. The parenthetical expressions on the left-hand side are now perfect squares.

$$(x^2 + 4x + 4) + (y^2 - 6y + 9) = 3 + 4 + 9$$

$$(x + 2)^2 + (y - 3)^2 = 16$$

Write each quadratic as a squared linear expression.

The center is (-2, 3) and the radius is $r = 4$.

Example 20: Sketch the graph of the equation $x^2 + y^2 + 2x - 6y + 7 = 0$ by first showing that it represents a circle and then finding its center and radius.

SOLUTION We first group the x-terms and y-terms as follows:

$$(x^2 + 2x) + (y^2 - 6y) = -7$$

Then we complete the square within each grouping, adding the appropriate constants to both sides of the equation:

$$(x^2 + 2x + 1) + (y^2 - 6y + 9) = -7 + 1 + 9$$

or

$$(x + 1)^2 + (y - 3)^2 = 3$$

Comparing this equation with the standard equation of a circle, we see that $h = -1$, $k = 3$ and $r = \sqrt{3}$, so the given equation represents a circle with center (-1, 3) and radius $r = \sqrt{3}$. It is sketched in Figure 1.18.

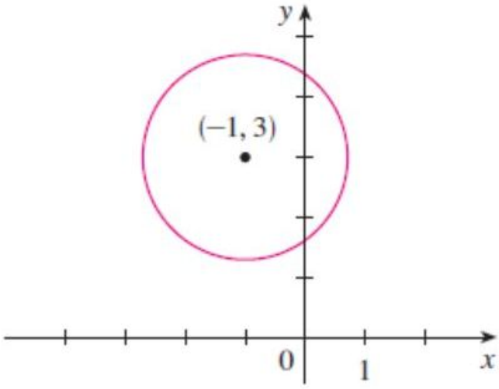


Figure 1.18

(b)Parabola

The geometric properties of parabolas will be reviewed later. Here we regard a parabola as a graph of an equation of the form $y = ax^2 + bx + c$.

Example 21: Draw the graph of the parabola $y = x^2$

Solution:

We set up a table of values, plot points, and join them by a smooth curve to obtain the graph in Figure 1.19.

x	$y = x^2$
0	0
$\pm\frac{1}{2}$	$\frac{1}{4}$
± 1	1
± 2	4
± 3	9

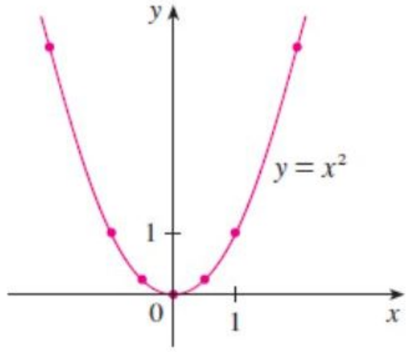


Figure 1.19

Figure 1.20 shows the graphs of several parabolas with equations of the form for various values of the number a. In each case the *vertex,* the point where the parabola changes direction, is the origin.

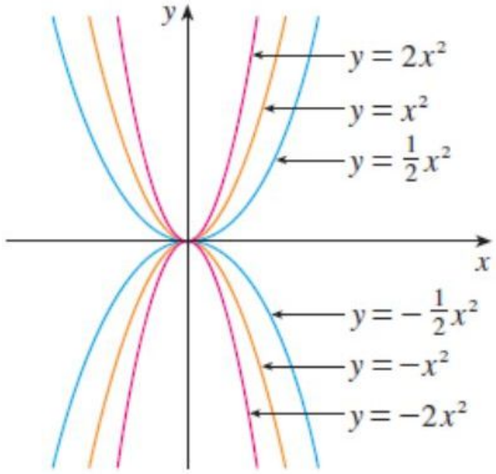


Figure 1.20

We see that:
The parabola $y = ax^2$ opens upward if $a > 0$
The parabola $y = ax^2$ opens downward if $a < 0$. (as in Figure 1.21)

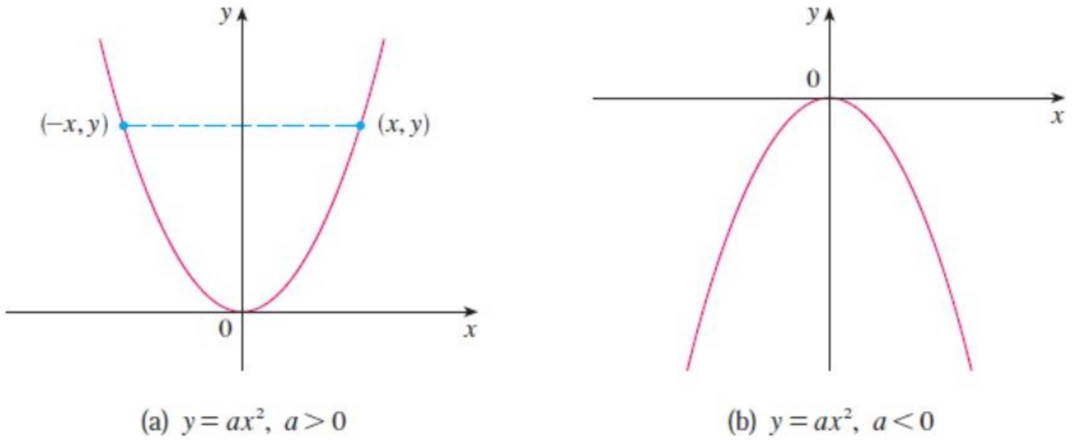


Figure 1.21

Note that:
The graph of an equation is symmetric with respect to the *y*-axis if the equation is unchanged when x is replaced by – *x*, and
The larger the value of | *a* | the narrower the parabola

Generally,

The Graph of $y = ax^2 + bx + c, \quad a \neq 0$
The graph of the equation $y = ax^2 + bx + c, a \neq 0$, is a parabola. The parabola opens upward if $a > 0$ and downward if $a < 0$. The **axis** is the line

$$x = -\frac{b}{2a}. \tag{2}$$

The **vertex** of the parabola is the point where the axis and parabola intersect. Its *x*-coordinate is $x = -b/2a$; its *y*-coordinate is found by substituting $x = -b/2a$ in the parabola's equation.

Example 22: Graph the equation $-\frac{1}{2}x^2 - x + 4$

Solution: Comparing the equation $y = ax^2 + bx + c$ with we see that

$$a = -\tfrac{1}{2}, \quad b = -1, \quad c = 4$$

Since $a < 0$ the parabola opens downward. From Equation (2) the axis is the vertical line

$$x = -\frac{b}{2a} = -\frac{(-1)}{2\left(-\frac{1}{2}\right)} = -1$$

When $x = -1$, we have

$$y = -\frac{1}{2}(-1)^2 - (-1) + 4 = \frac{9}{2}$$

The vertex is (-1, 9/2).
The x-intercepts are where $y = 0$:

$$-(1/2)\,x^2 - x + 4 = 0$$
$$x^2 + 2x - 8 = 0$$
$$(x - 2)\,(x + 4) = 0$$
$$x = 2,\; x = -4$$

We plot some points, sketch the axis, and use the direction of opening to complete the graph in Figure 1.22

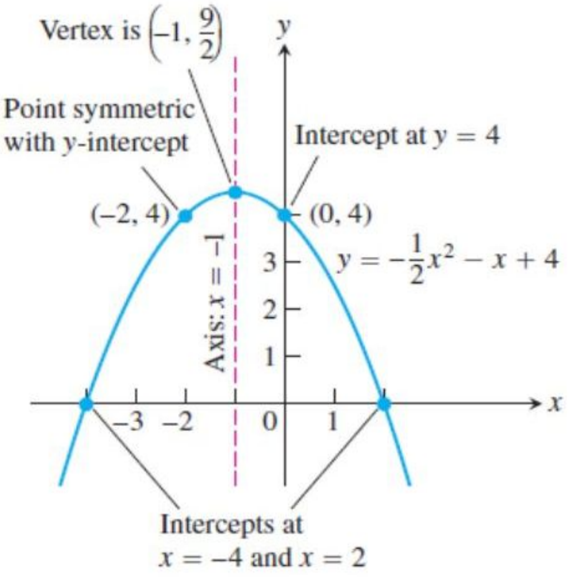
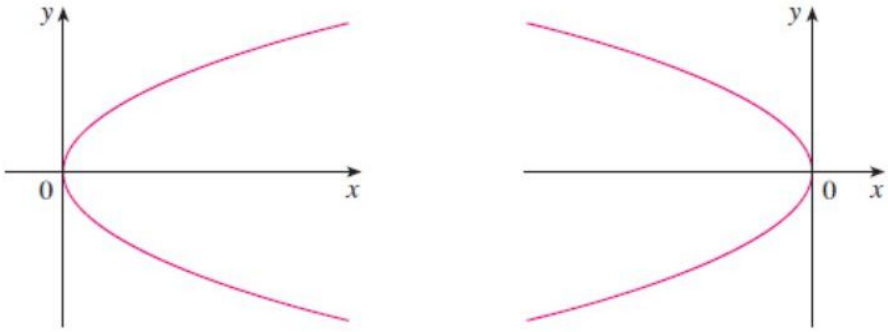


Figure 1.22

If we interchange x and y in the equation $y = ax^2$, the result is $x = ay^2$, which also represents a parabola. The parabola $x = ay^2$ opens to the right if $a > 0$ and to the left if $a < 0$. (See Figure 1.23). This time the parabola is symmetric with respect to the x-axis because if (x, y) satisfies $x = ay^2$, then so does $(x, -y)$.



(a) $x = ay^2,\ a > 0$ (b) $x = ay^2,\ a < 0$

Figure 1.23

> The graph of an equation is symmetric with respect to the x-axis if the equation is unchanged when y is replaced by $-y$.

(c)ELLIPSES

The curve with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a and b are positive numbers, is called an ellipse in standard position.

The most important properties of the ellipses are:

- The ellipse is symmetric with respect to both axes, i.e the above Equation is unchanged if x is replaced by $-x$ or y is replaced by $-y$.
- The ***x*-intercepts** of a graph are the x-coordinates of the points where the graph intersects the x-axis. They are found by setting $y = 0$ in the equation of the graph.

- The ***y*-intercepts** of a graph are the y-coordinates of the points where the graph intersects the y-axis. They are found by setting $x = 0$ in the equation of the graph. See Figure 1.24

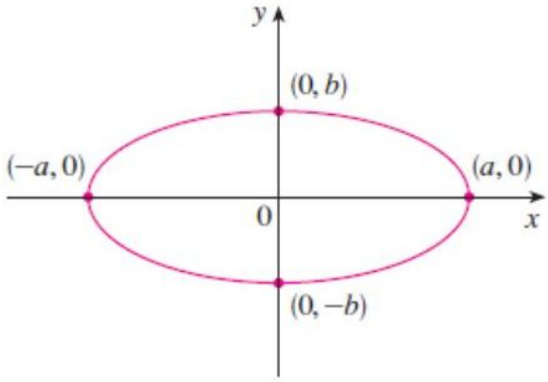


Figure 1.24

Example 23: Sketch the graph of $9x^2 + 16y^2 = 144$.
Solution: We divide both sides of the equation by 144:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

The equation is now in the standard form for an ellipse, so we have $a^2 = 16$, $b^2 = 9$, $a = 4$ and $b = 3$. The x-intercepts are ± 4; the y-intercepts are ± 3. The graph is sketched in Figure 1.25.

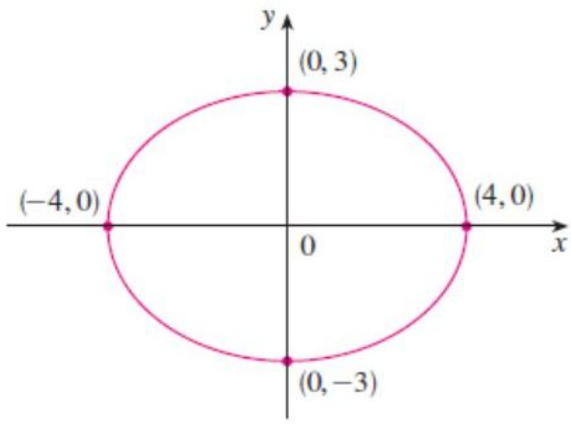


Figure 1.25

(d)HYPERBOLAS

The curve with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where a and b are positive numbers, is called a hyperbola in standard position.
The most important properties of the hyperbola are:

- The hyperbola is symmetric with respect to both axes, i.e the above Equation is unchanged if x is replaced by $-x$ or y is replaced by $-y$.
- The ***x*-intercepts** of a graph are the x-coordinates of the points where the graph intersects the x-axis. They are found by setting $y = 0$ in the equation of the graph: $y = 0$ obtain $x^2 = a^2$ and $x = \pm a$.
- If we put $x = 0$ in Equation 3, we get $y^2 = - b^2$, which is impossible, so there is no y-intercept.
- The hyperbola consists of two parts, called its branches.
- The hyperbola have two asymptotes, which are the lines $y = (b/a)x$ and $y = -(b/a)x$ shown in Figure 1.26. Both branches of the hyperbola approach the asymptotes; that is, they come arbitrarily close to the asymptotes. This involves the idea of a limit, which is discussed in proceeding chapters.

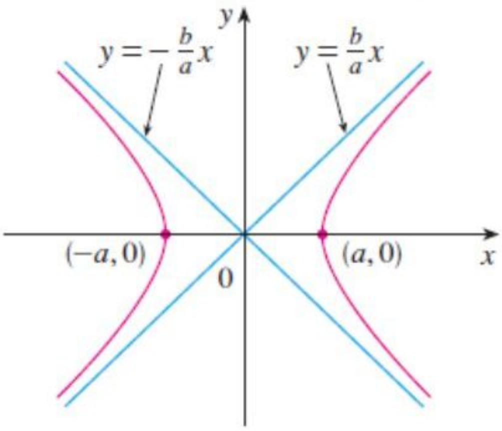


Figure 1.26

By interchanging the roles of and we get an equation of the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

which also represents a hyperbola and is sketched in Figure 1.27.

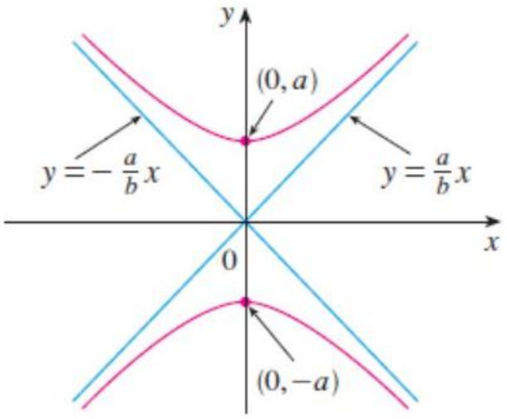


Figure 1.27

Example 24: Sketch the curve $9x^2 - 4y^2 = 36$.
Solution: Dividing both sides by 36, we obtain:

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

which is the standard form of the equation of a hyperbola. Since $a^2 = 4$, the x-intercepts are ± 2. Since $b^2 = 9$, we have b = 3 and the asymptotes are $y = \pm (3/2)\, x$. The hyperbola is sketched in Figure 1.28.

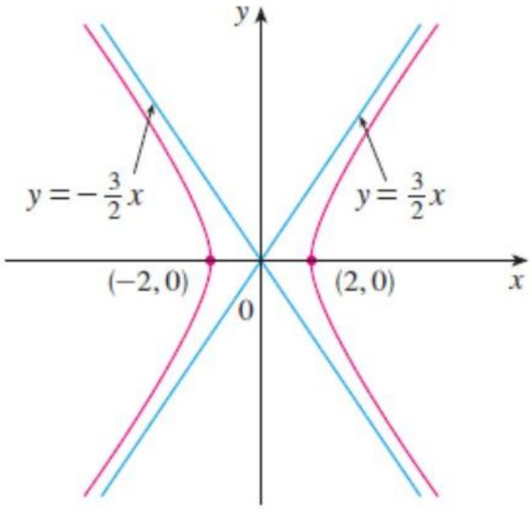


Figure 1.28

If $b = a$, a hyperbola has the equation $x^2 - y^2 = a^2$ (or $y^2 - x^2 = a^2$) and is called an *equilateral hyperbola* [see Figure 1.29(a)]. Its asymptotes are $y = \pm x$, which are perpendicular.
If an equilateral hyperbola is rotated by 45^0, the asymptotes become the x- and y-axes, and it can be shown that the new equation of the hyperbola is $xy = k$, where k is a constant [see Figure 1.29(b)]

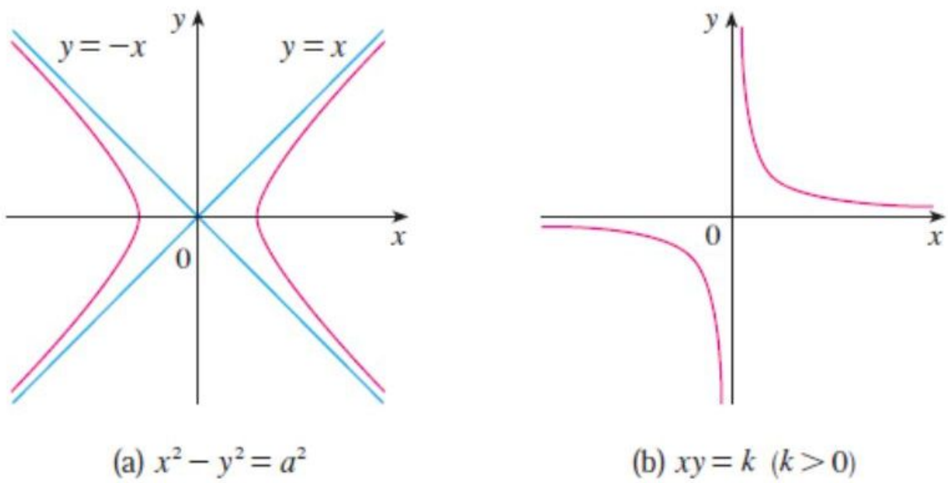


Figure 1.29

1.3 Functions; Domain and Range

Functions arise whenever one quantity depends on another. A function can be represented by an equation, a graph, a numerical table, or a verbal description.

A **function** f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

We usually consider functions for which the sets D and E are sets of **real numbers**.
The set D is called the **domain** of the function.
The number $f(x)$ is the **value of** f **at** x and is read "f of x."
The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain.
A symbol that represents an arbitrary number in the *domain* of a function f is called an **independent variable**.
A symbol that represents a number in the *range* of f is called a **dependent variable**.
Thus we can think of the **domain** as the set of all possible inputs and the **range** as the set of all possible outputs if we see the function as a kind of machine (Figure 1.30).

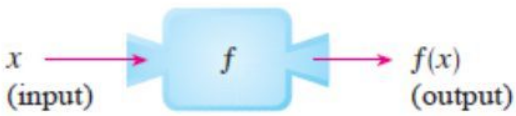


Figure 1.30

Example 25: Verify the domains and associated ranges of the following functions.

(a) $y = x^2$

The formula $y = x^2$ gives a real y-value for any real number x, so the **domain** is $(-\infty, \infty)$. The **range** of $y = x^2$ is $[0, \infty)$ because the square of any real number is nonnegative and every nonnegative number y is the square of its own square root.

(b) $y = 1/x$

The formula $y = 1/x$ gives a real y-value for every x except $x = 0$. For consistency in the rules of arithmetic, **we cannot divide any number by zero**. The range of $y = 1/x$, the set of reciprocals of all nonzero real numbers, is the set of all nonzero real numbers, since $y = 1/(1/y)$. That is, for $y \neq 0$ the number $x = 1/y$ is the input assigned to the output value y.

(c) $y = \sqrt{x}$

The formula $y = \sqrt{x}$ gives a real y-value only if $x \geq 0$.
The **domain** of $y = \sqrt{x}$ is $[0, \infty)$
The **range** of $y = \sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number's square root.

(d) $y = \sqrt{4 - x}$
The quantity $4 - x$ cannot be negative. That is, $4 - x \geq 0$, or $x \leq 0$.
The formula gives real y-values for all $x \leq 4$. The **domain** is $(-\infty, 4]$
The **range** of function is $[0, \infty)$, the set of all nonnegative numbers.

(e) $y = \sqrt{1 - x^2}$
The **domain** is $[-1, 1]$
The **range** is $[0, 1]$

1.3.1 Graphs of Functions

The most common method for visualizing a function is its graph. If f is a function with domain D, then its graph is the set of ordered pairs

$$\{(x, f(x)) \mid x \in D\}$$

(Notice that these are input-output pairs.) In other words, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f.

Example 26: Graph the function $y = x^2$ over the interval [-2, 2].

Solution:

1. Make a table of *xy*-pairs that satisfy the equation $y = x^2$.

x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

2. Plot the points (*x*, *y*) whose coordinates appear in the table (see Figure 1.31)

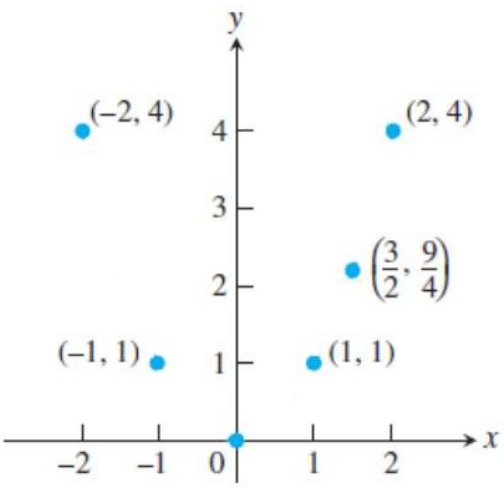


Figure 1.31

3. Draw a *smooth* curve (labeled with its equation) through the plotted points. (Figure 1.32)

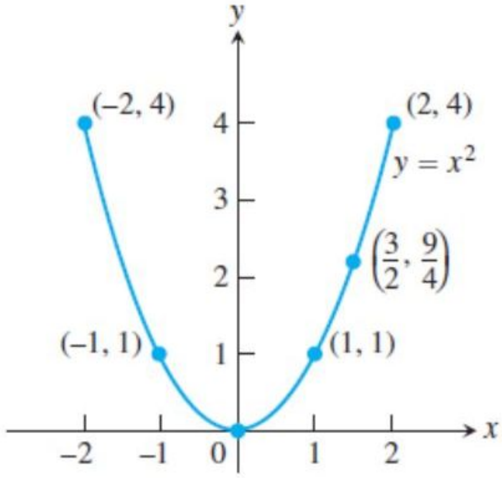


Figure 1.32

1.3.2 Piecewise-Defined Functions

Sometimes a function is described by using different formulas on different parts of its domain. One example is the **absolute value function.**

$$|\boldsymbol{x}| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0, \end{cases}$$

whose graph is given in Figure 1.33. The right-hand side of the equation means that the function equals x if $x \geq 0$, and equals - x if $x < 0$.

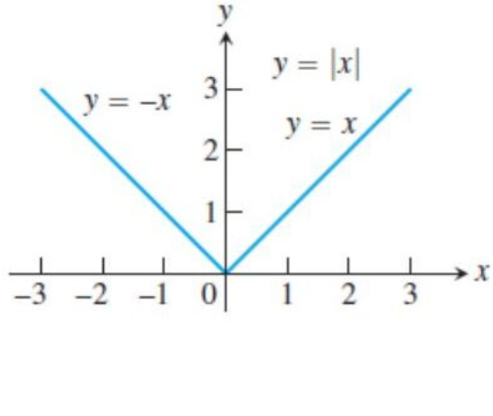


Figure 1.33

Example 27: The function

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

The values of f are given by:
$y = -x$ when $x < 0$,
$y = x^2$ when $0 \leq x \leq 1$ and
$y = 1$ when $x > 1$

The function, however, is *just one function* whose domain is the entire set of real numbers (see Figure 1.34)

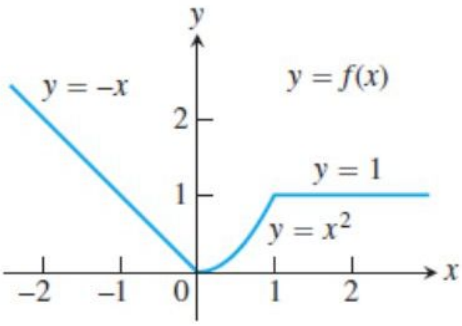


Figure 1.34

Example 28: A function is defined by

$$f(x) = \begin{cases} 1 - x, & x \geq 0 \\ x^2, & x < 0, \end{cases}$$

Evaluate $f(0)$, $f(1)$and $f(2)$ and sketch the graph.
Solution:

Since $0 \leq 1$, we have $f(0) = 1 - 0 = 1$
Since $1 \leq 1$, we have $f(1) = 1 - 1 = 0$
Since $2 > 1$, we have $f(2) = 2^2 = 4$

See Figure 1.35

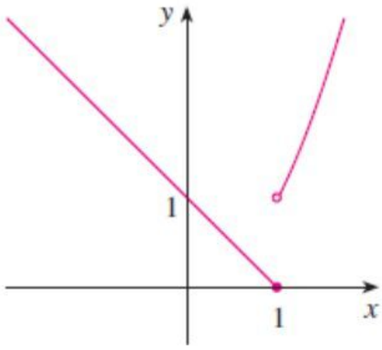


Figure 1.35

1.3.3 Increasing and Decreasing Functions

The graph shown in Figure 1.36 rises from A to B, falls from B to C, and rises again from C to D. The function *f* is said to be increasing on the interval [a, b], decreasing on [b, c], and increasing again on [c, d].

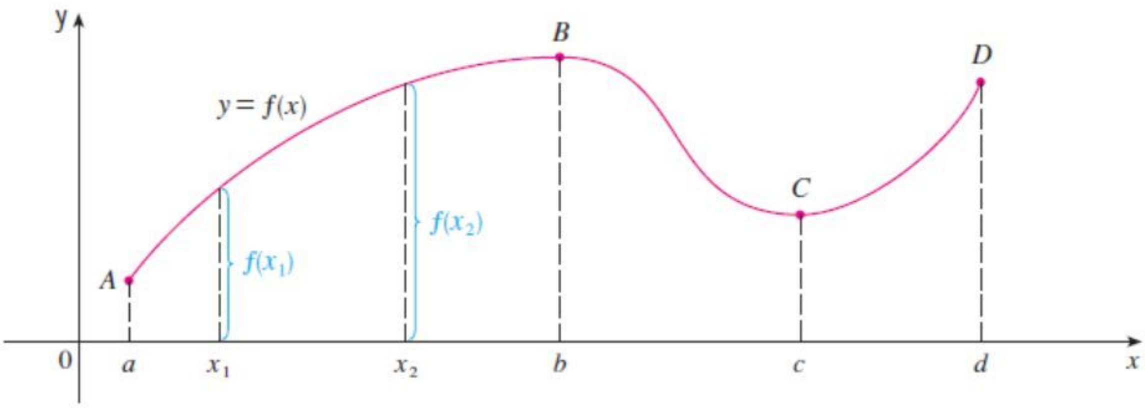


Figure 1.36

DEFINITIONS Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

Example 29: investigate the increasing and decreasing intervals of the functions $y = x^2$

Solution:

In the definition of an increasing function it is important to realize that the inequality $f(x_1) < f(x_2)$ must be satisfied for *every* pair of numbers x_1 and x_2 in I with $x_2 < x_1$.

For interval $[0, \infty)$

$f(1) = 1$

$f(2) = 4$

So that, according to the 1st definition the function is increasing on the interval $[0, \infty)$

For interval $(- \infty, 0]$

$f(-1) = 1$

$f(-2) = 4$

The function is decreasing on the interval $(- \infty, 0]$

See Figure 1.37

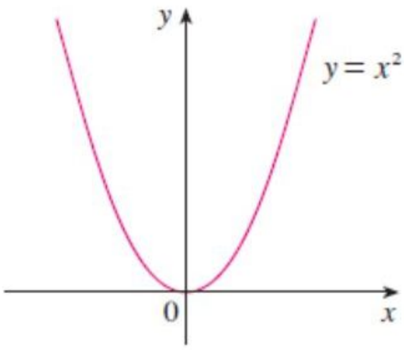


Figure 1.37

1.3.4 Even Functions and Odd Functions: Symmetry

The graphs of *even* and *odd* functions have characteristics symmetry properties.

DEFINITIONS	A function $y = f(x)$ is an **even function of** x if $f(-x) = f(x)$, **odd function of** x if $f(-x) = -f(x)$, for every x in the function's domain.

Generally,
The graph of an even function is **symmetric about the *y*-axis**. As for the function $f(x) = x^2$ (see Figure 1.38)
Since always $f(-x) = f(x)$

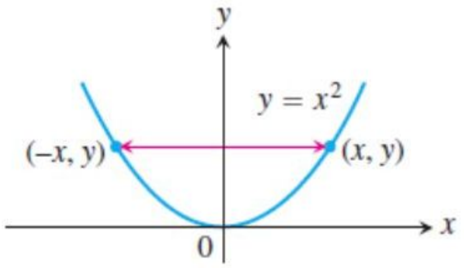


Figure 1.38

The graph of an odd function is **symmetric about the origin**.
For example the function $y = x^3$ (Figure 1.39)
Always $f(-x) = - f(x)$

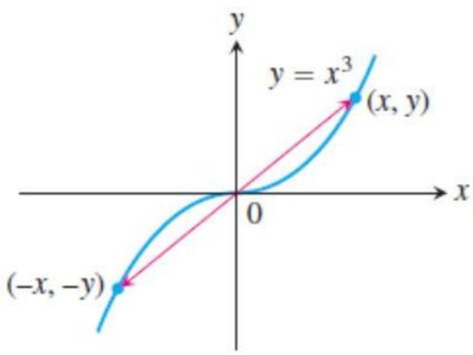


Figure 1.39

Example 30: Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ (b) $g(x) = 1 - x^4$ (c) $h(x) = 2x - x^2$

Solution:

(a)

$$f(-x) = (-x)^5 + (-x) = (-1)^5 x^5 + (-x)$$
$$= -x^5 - x = -(x^5 + x)$$
$$= -f(x)$$

Therefore is an odd function.

(b)

$$g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$$

so g is even

(c)

$$h(-x) = 2(-x) - (-x)^2 = -2x - x^2$$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd.

The graphs of the functions are shown in Figure 1.40. Notice that the graph of h is symmetric neither about the y-axis nor about the origin.

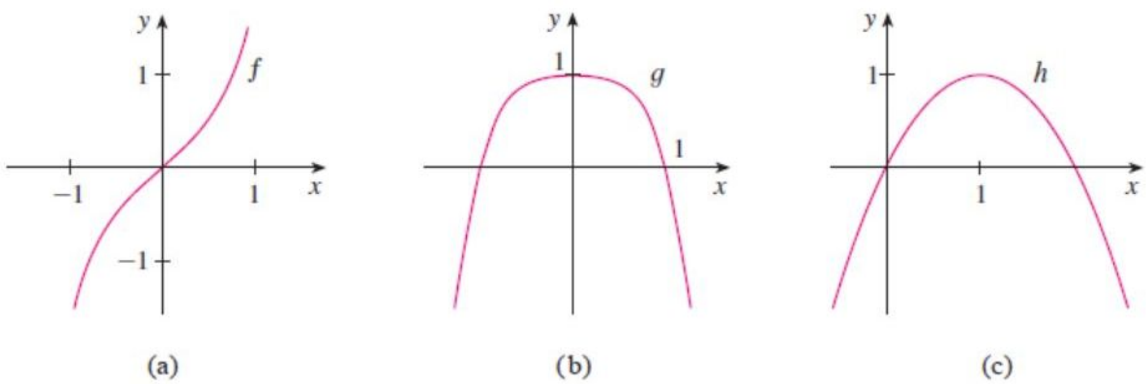


Figure 1.40

1.3.5 Trigonometric Functions

(a) Angles

Angles can be measured in degrees or in radians (abbreviated as rad). The angle given by a complete revolution contains 360^o, which is the same as 2π rad. Therefore

$$\pi \text{ rad} = 180°$$

and

$$1 \text{ rad} = \left(\frac{180}{\pi}\right)^0 \approx 57.3^o$$
$$1^o = \left(\frac{\pi}{180}\right)^0 \text{rad} \approx 0.017^o$$

Example 31:
(a) Find the radian measure of 60^0.
(b) Express $5\pi/4$ rad in degrees.

Solution:
(a) From above equations we see that to convert from degrees to radians we multiply by $\pi/180$. Therefore

$$60° = 60\left(\frac{\pi}{180}\right) = \frac{\pi}{3} \text{rad}$$

(b) To convert from radians to degrees we multiply by $180/\pi$. Thus

$$\frac{5\pi}{4} rad = \frac{5\pi}{4}\left(\frac{180}{\pi}\right) = 225°$$

Table 1.2 shows the equivalence between degree and radian measures for some basic angles.

TABLE 1.2 Angles measured in degrees and radians

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

Figure 1.41 shows a sector of a circle with central angle θ and radius r subtending an arc with length a.

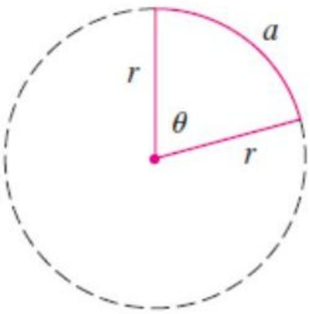


Figure 4.41

Since the length of the arc is proportional to the size of the angle, and since the entire circle has circumference $2\pi r$ and central angle 2π, we have:

$$\frac{\theta}{2\pi} = \frac{a}{2\pi r}$$

Solving this equation for θ and for a, we obtain

$\theta = \frac{a}{r}$	$a = r\theta$

Remember that the above equations are valid only when is measured in **radians**.

Example 32:
(a) If the radius of a circle is 5 cm, what angle is subtended by an arc of 6 cm?
(b) If a circle has radius 3 cm, what is the length of an arc subtended by a central angle of 3π/8 rad?
Solution:

(a) Using Equation $\theta = \frac{a}{r}$ and $a = r\theta$ with $a = 6$ and $r = 5$, we see that the angle is

$$\theta = 6/5 = 1.2\ rad$$

(b) With $r = 3$ cm and $\theta = 3\pi/8$ rad, the arc length is:

$$a = r\theta = 3\left(\frac{3\pi}{8}\right) = \frac{9\pi}{8}\text{ cm}$$

(b)standard position

The **standard position** of an angle occurs when we place its vertex at the origin of a coordinate system and its initial side on the positive x-axis as in Figure 4.42.

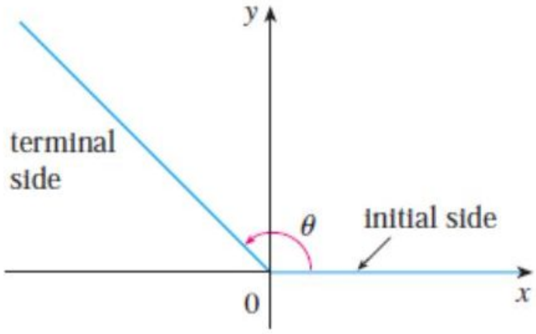


Figure 4.42: $\theta \geq 0$

A **positive** angle is obtained by rotating the initial side **counterclockwise** until it coincides with the terminal side. (as in Figure 4.42)
A negative angle are obtained by **clockwise** rotation as in Figure 4.43

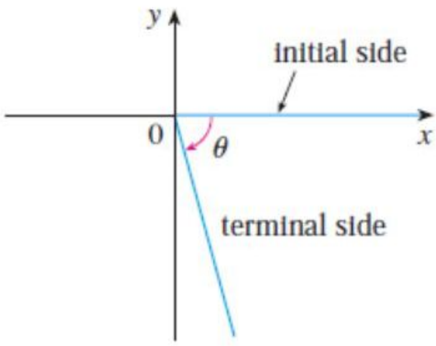


Figure 4.43: $\theta < 0$

Figure 4.44 shows several examples of angles in standard position. Notice that different angles can have the same terminal side. For instance, the angles $3\pi/4$, $-5\pi/4$ and $11\pi/4$ have the same initial and terminal sides because:

$$\frac{3\pi}{4} - 2\pi = -\frac{5\pi}{4} \qquad \frac{3\pi}{4} + 2\pi = \frac{11\pi}{4}$$

and 2π rad represents a complete revolution.

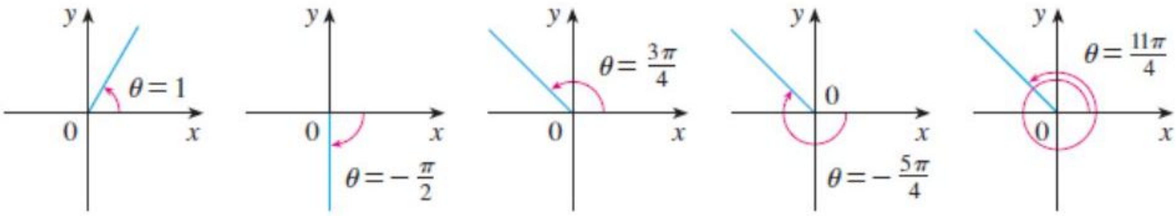


Figure 4.44

(c)The Six Basic Trigonometric Functions

For an acute angle θ the six trigonometric functions are defined as ratios of lengths of sides of a right triangle as follows (see Figure 1.45).

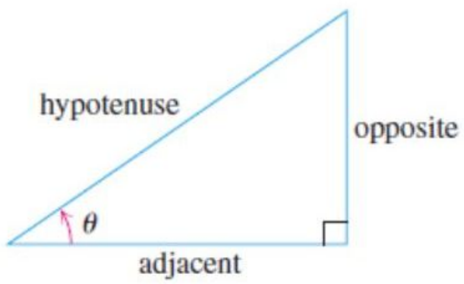


Figure 1.45

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

We extend this definition to obtuse and negative angles by first placing the angle in standard position in a circle of radius *r*.
We then define the trigonometric functions in terms of the coordinates of the point *P*(*x*, *y*) where the angle's terminal ray intersects the circle (Figure 1.46).

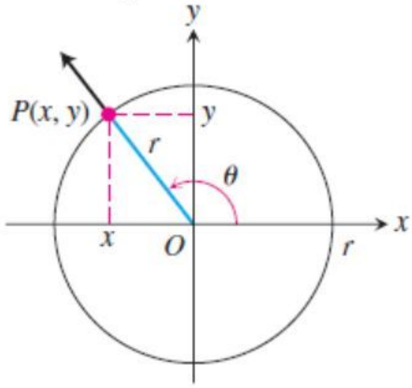


Figure 1.46

$$\textbf{sine:}\quad \sin\theta = \frac{y}{r} \qquad \textbf{cosecant:}\quad \csc\theta = \frac{r}{y}$$

$$\textbf{cosine:}\quad \cos\theta = \frac{x}{r} \qquad \textbf{secant:}\quad \sec\theta = \frac{r}{x}$$

$$\textbf{tangent:}\quad \tan\theta = \frac{y}{x} \qquad \textbf{cotangent:}\quad \cot\theta = \frac{x}{y}$$

These extended definitions agree with the right-triangle definitions when the angle is **acute**.
Notice also that whenever the quotients are defined,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

As you can see, ***tan θ*** and ***sec θ*** are not defined if $x = \cos\theta = 0$.
This means they are not defined if θ is $\pm\pi/2, \pm 3\pi/2, \ldots$
Similarly, ***cot θ*** and **csc *θ*** are not defined for values of θ for which y = 0, namely $\theta = 0, \pm\pi, \pm 2\pi, \ldots$
The exact trigonometric ratios for certain angles can be read from the triangles in Figure 1.47. For instance,

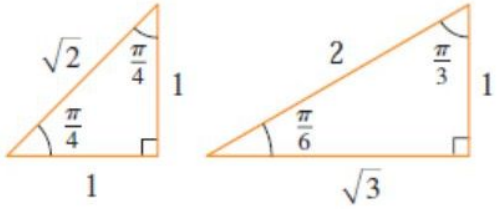


Figure 1.47

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

The signs of the trigonometric functions for angles in each of the four quadrants can be remembered by means of the rule "All Students Take Calculus" or "CAST" rule shown in Figure 1.48.

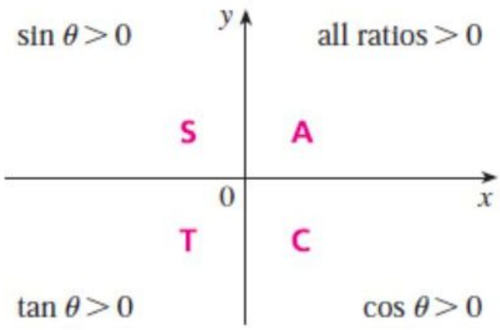


Figure 1.48

Example 33: Find the exact trigonometric ratios for $\theta = 2\pi/3$.
Solution: From the triangle in Figure 1.49 we see that:

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}, \qquad \cos\frac{2\pi}{3} = -\frac{1}{2}, \qquad \tan\frac{2\pi}{3} = -\sqrt{3}$$

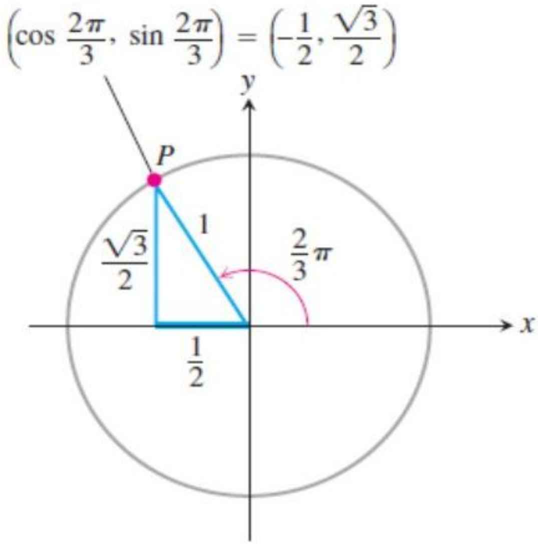


Figure 1.49

Using a similar method we determined the values of sin θ, cos θ, and tan θ shown in Table 1.3

Table 1.3

Degrees	**−180**	**−135**	**−90**	**−45**	**0**	**30**	**45**	**60**	**90**	**120**	**135**	**150**	**180**	**270**	**360**
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
sin θ	0	$\frac{-\sqrt{2}}{2}$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0
cos θ	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	−1	0	1
tan θ	0	1		−1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	−1	$\frac{-\sqrt{3}}{3}$	0		0

Example 34: If cos $\theta = 2/5$ and $0 < \theta < \pi/2$, find the other five trigonometric functions of θ.

Solution:

Since cos θ =2/5, we can label the hypotenuse as having length 5 and the adjacent side as having length 2 in Figure 1.50. If the opposite side has length x, then the Pythagorean Theorem gives $x^2 + 4 = 25$ and so $x^2 = 21$, $x = \sqrt{21}$.

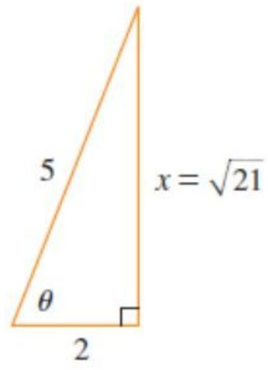


Figure 1.50

We can now use the diagram to write the other five trigonometric functions:

$$\sin\theta = \frac{\sqrt{21}}{5} \qquad \tan\theta = \frac{\sqrt{21}}{2}$$

$$\csc\theta = \frac{5}{\sqrt{21}} \qquad \sec\theta = \frac{5}{2} \qquad \cot\theta = \frac{2}{\sqrt{21}}$$

Example 35: Use a calculator to approximate the value of *x* in Figure 1.51.

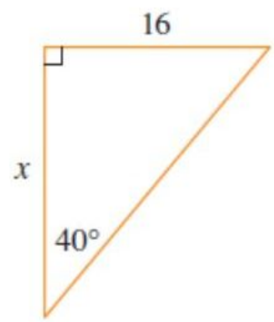


Figure 1.51

Solution: From the diagram we see that:

$$\tan 40° = \frac{16}{x}$$

Therefore,

$$x = \frac{16}{\tan 40°} \approx 19.07$$

(d)Trigonometric Identities

A trigonometric identity is a relationship among the trigonometric functions. The most elementary are the following:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} \qquad (1)$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} \qquad (2)$$

For the next identity we refer back to Figure 1.46. The distance formula (or, equivalently, the Pythagorean Theorem) tells us that $x^2 + y^2 = r^2$. Therefore

$$sin^2\theta + cos^2\theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$$

We have therefore proved one of the most useful of all trigonometric identities:

$$\cos^2\theta + \sin^2\theta = 1. \qquad (3)$$

This equation, true for all values of θ, is the most frequently used identity in trigonometry.
Dividing this identity in turn by $\cos^2\theta$ and $\sin^2\theta$ gives:

$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

The identity

$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$

Show that **sin** is an odd function and **cos** is an even function.

Since the angles θ and $\theta + 2\pi$ have the same terminal side, we have:

$$\sin(\theta + 2\pi) = \sin\theta \qquad \cos(\theta + 2\pi) = \cos\theta$$

The remaining trigonometric identities are all consequences of two basic identities called the **addition formulas**:

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

By substituting – *y* for in above equations and using equations [*sin (-θ) = -sin (θ)* and *cos (-θ) = cos (θ)*] we obtain the following **subtraction formulas**:

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

Then, by dividing the formulas in **addition formulas** or **subtraction formulas**, we obtain the corresponding formulas for *tan (x ± y)*:

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$
$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

If we put $y = x$ in the addition formulas, we get the **double-angle formulas**:

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

Then, by using the identity $\sin^2 x + \cos^2 x = 1$, we obtain the following alternate forms of the double-angle formulas for $\cos 2x$:

$$\cos 2x = 2\cos^2 x - 1$$

$$\cos 2x = 1 - 2\sin^2 x$$

If we now solve these equations for $\cos^2 x$ and $\sin^2 x$, we get the following **half-angle formulas**, which are useful in integral calculus:

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

Finally, we state the **product formulas**, which can be deduced from **addition and subtraction formulas**:

$$\sin x \cos y = \tfrac{1}{2}[\sin(x + y) + \sin(x - y)]$$

$$\cos x \cos y = \tfrac{1}{2}[\cos(x + y) + \cos(x - y)]$$

$$\sin x \sin y = \tfrac{1}{2}[\cos(x - y) - \cos(x + y)]$$

Example 36 Find all values of x in the interval $[0, 2\pi]$ such that $\sin x = \sin 2x$.
Solution: Using the double-angle formula, we rewrite the given equation as

$$\sin x = 2 \sin x \cos x \qquad \text{or} \qquad \sin x(1 - 2\cos x) = 0$$

Therefore, there are two possibilities:

$$\sin x = 0 \quad \text{or} \quad 1 - 2\cos x = 0$$

$$x = 0, \pi, 2\pi \qquad \cos x = ½$$

$$x = \pi/3, 5\pi/3$$

The given equation has five solutions:0, π/3, π, 5π/3, and 2π.

(e)Periodicity and Graphs of the Trigonometric Functions

The graph of the trigonometric function is obtained by plotting points for one period and then using the periodic nature of the function to complete the graph.

DEFINITION A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

We describe this repeating behavior by saying that the six basic trigonometric functions are *periodic:*

Periods of Trigonometric Functions

Period π: $\tan(x + \pi) = \tan x$
$\cot(x + \pi) = \cot x$

Period 2π: $\sin(x + 2\pi) = \sin x$
$\cos(x + 2\pi) = \cos x$
$\sec(x + 2\pi) = \sec x$
$\csc(x + 2\pi) = \csc x$

Example 37: plot the trigonometric function ***sin x***.

Solution:
The function will be: $y = f(x) = \sin(x)$
Domain: $-\infty < x < \infty$
Range: $-1 \leq y \leq 1$

Make a table for xy values

x	***y = sin (x)***
0	0
π/2	1
π	0
3 π/2	-1
2 π	0
- π/2	-1
- π	0

The plot of *sin x* was shown in Figure 1.52. The function *y = sin x* is an odd function (*f (-x) = - f (x)*)

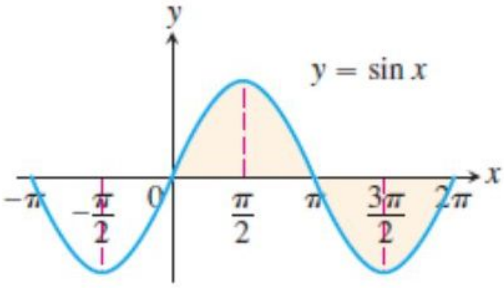
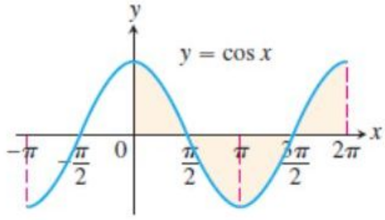
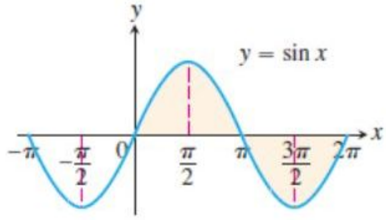


Figure 1.52

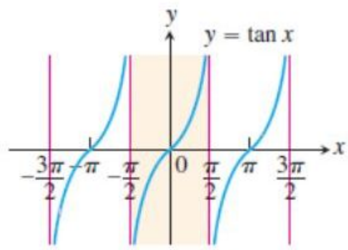
Figure 1.53 shows the graphs of the six basic trigonometric functions using radian measure. The shading for each trigonometric function indicates its periodicity.



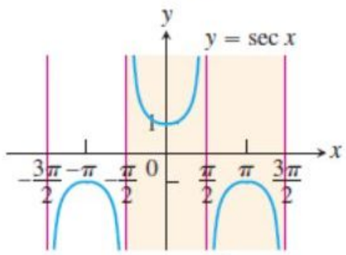
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π
(a)



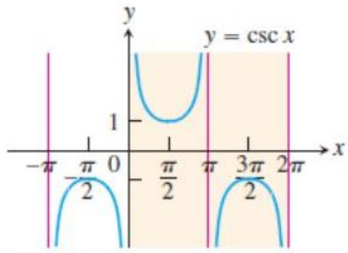
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π
(b)



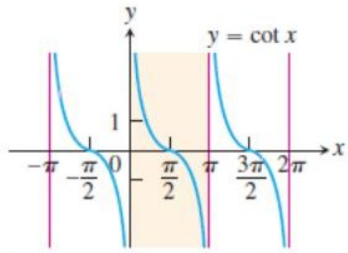
Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $-\infty < y < \infty$
Period: π
(c)



Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $y \le -1$ or $y \ge 1$
Period: 2π
(d)



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $y \le -1$ or $y \ge 1$
Period: 2π
(e)



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $-\infty < y < \infty$
Period: π
(f)

CHAPTER 2

Limits and Continuity

2.1 Limits

If the values of $f(x)$ tend to get closer and closer to the number L as x gets closer and closer to the number a (from either side of a) but $x \neq a$.

> **I DEFINITION** We write
>
> $$\lim_{x \to a} f(x) = L$$
>
> and say "the limit of $f(x)$, as x approaches a, equals L"
>
> if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

Example 1: Find the limit of the function $f(x) = x^2 - x + 2$ as x approaches 2.

Solution:

Let's investigate the behavior of the function $f(x) = x^2 - x + 2$ for values of x near 2. The following table gives values of $f(x)$ for values of x close to 2, but not equal to 2.

x	f(x)	x	f(x)
1.0	2.000000	3.0	8.000000
1.5	2.750000	2.5	5.750000
1.8	3.440000	2.2	4.640000
1.9	3.710000	2.1	4.310000
1.95	3.852500	2.05	4.152500
1.99	3.970100	2.01	4.030100
1.995	3.985025	2.005	4.015025
1.999	3.997001	2.001	4.003001

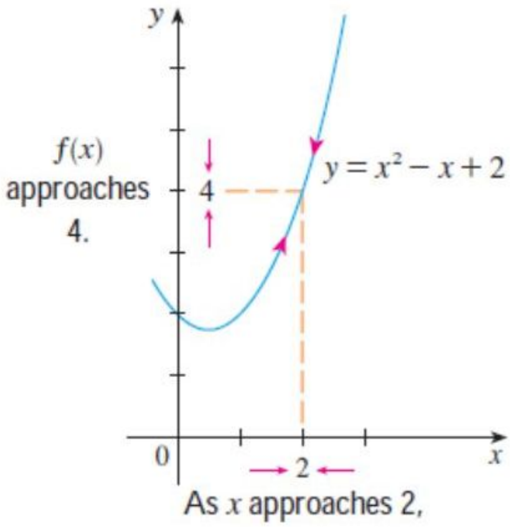


Figure 1

From the table and the graph of *f* (a parabola) shown in Figure 1 we see that when *x* is close to 2 (on either side of 2), *f (x)* is close to 4. In fact, it appears that we can make the values of *f (x)* as close as we like to 4 by taking *x* sufficiently close to 2. We express this by saying "the limit of the function $f(x) = x^2 - x + 2$ as *x* approaches 2 is equal to 4."
The notation for this is:

$$\lim_{x\to 2}(x^2 - x + 2) = 4$$

Example 2: Guess the value of $\lim_{x\to 1}\frac{x-1}{x^2-1}$
Solution:
Notice that the function $f(x) = (x - 1) / (x^2 - 1)$ is not defined when $x = 1$, but that doesn't matter because the definition of $\lim_{x\to a} f(x)$ says that we consider values of *x* that are close to *a* but not equal to *a*.
The tables below give values of *f (x)* (correct to six decimal places) for values of *x* that approach 1 (but are not equal to 1).

x < 1	f(x)
0.5	0.666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025

x > 1	f(x)
1.5	0.400000
1.1	0.476190
1.01	0.497512
1.001	0.499750
1.0001	0.499975

On the basis of the values in the tables, we make the guess that

$$\lim_{x\to 1}\frac{x-1}{x^2-1} = 0.5$$

The value of $\lim_{x\to 1}\frac{x-1}{x^2-1}$ can be solved to give same results by:

$$\lim_{x\to 1}\frac{x-1}{x^2-1} = \lim_{x\to 1}\frac{(x-1)}{(x-1)(x+1)}$$

$$\lim_{x\to 1}\frac{1}{x+1}$$

$$\lim_{x\to 1}\frac{1}{(1)+1} = 0.5$$

Example 2 is illustrated by the graph of in Figure 2.

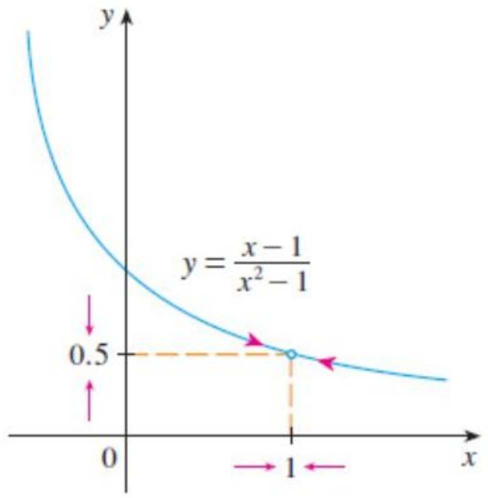


Figure 2

Example: 3

(a) If f is the **identity function** $f(x) = \mathbf{x}$, then for any value of x_0 (Figure 3a),

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0$$

(b) If f is the **constant function** $f(x) = k$ (function with the constant value k), then for any value of x_0 (Figure 3b),

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} k = k$$

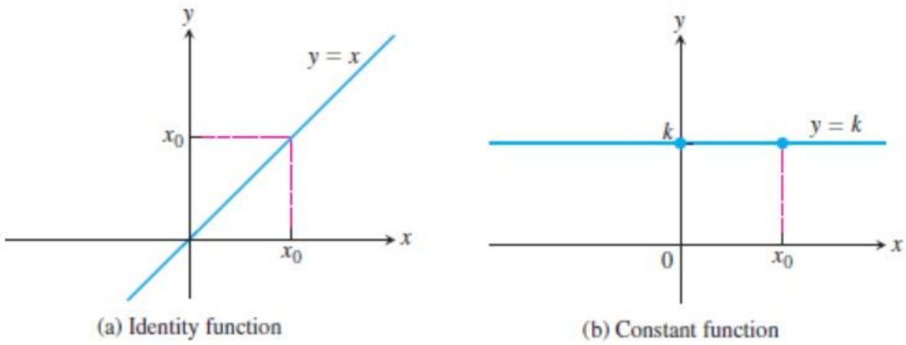


Figure 3

Example 4: Find $\lim_{x \to 0} \frac{1}{x^2}$ if it exists.

Solution: As x becomes close to 0, x^2 also becomes close to 0, and $1/x^2$ becomes very large. (See the table).

x	$\frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10,000
± 0.001	1,000,000

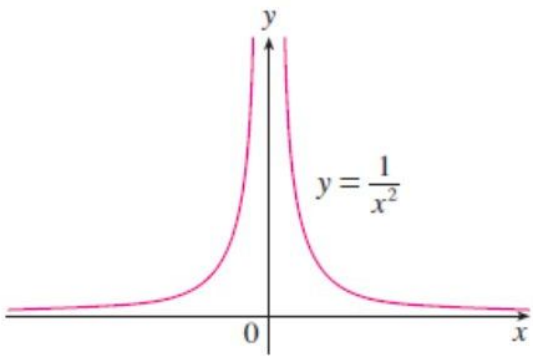


Figure 4

In fact, it appears from the graph of the function $f(x) = 1/x^2$ shown in Figure 4 that the values of $f(x)$ can be made arbitrarily large by taking x close enough to 0. Thus the values of $f(x)$ do not approach a number, so $\lim_{x\to 0}(1/x^2)$ **does not exist**.
To indicate the kind of behavior exhibited in Example 4, we use the notation:

$$\lim_{x\to 0}\frac{1}{x^2} = \infty$$

Example 5: Investigate $\lim_{x\to 0} \sin\frac{\pi}{x}$

Solution:

Again the function $f(x) = \sin(\pi/x)$ is undefined at 0. The graph of function was shown in Figure 5.

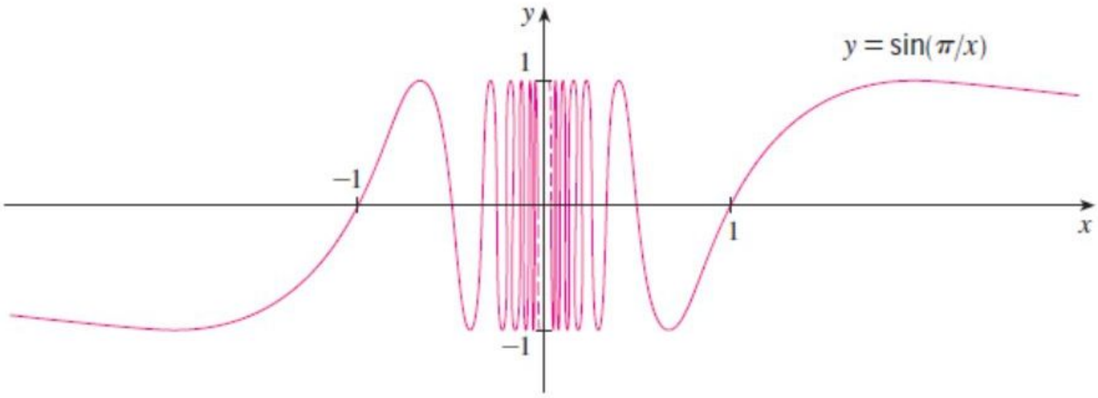


Figure 5

The graph indicate that the values of $\sin(\pi/x)$ oscillate between 1 and -1 infinitely often as x approaches 0.
Since the values of do not approach a fixed number as approaches 0,

$$\lim_{x\to 0} \sin\frac{\pi}{x} \quad \text{do not exist}$$

2.1.1 The Limit Laws

To calculate limits of functions that are arithmetic combinations of functions having known limits, we can use several easy rules:

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x \to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x \to c} f(x) = L > 0$.)

Example 6: Use the observations $\lim_{x \to c} k = k$ and $\lim_{x \to c} x = c$ (Example 3) and the properties of limits to find the following limits:

(a) $\lim_{x \to c}(x^3 + 4x^2 - 3)$ (b) $\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5}$ (c) $\lim_{x \to -2} \sqrt{4x^2 - 3}$

Solution:

(a) $\lim_{x \to c}(x^3 + 4x^2 - 3) = \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3$ Sum and Difference Rules

$= c^3 + 4c^2 - 3$ Power and Multiple Rules

(b) $\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \to c}(x^4 + x^2 - 1)}{\lim_{x \to c}(x^2 + 5)}$ Quotient Rule

$= \frac{\lim_{x \to c} x^4 + \lim_{x \to c} x^2 - \lim_{x \to c} 1}{\lim_{x \to c} x^2 + \lim_{x \to c} 5}$ Sum and Difference Rules

$= \frac{c^4 + c^2 - 1}{c^2 + 5}$ Power or Product Rule

(c) $\lim_{x \to -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x \to -2}(4x^2 - 3)}$ Root Rule with $n = 2$

$= \sqrt{\lim_{x \to -2} 4x^2 - \lim_{x \to -2} 3}$ Difference Rule

$= \sqrt{4(-2)^2 - 3}$ Product and Multiple Rules

$= \sqrt{16 - 3}$

$= \sqrt{13}$

THEOREM 2—Limits of Polynomials

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

THEOREM 3—Limits of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Example 7: The following calculation illustrates Theorems 2 and 3:

$$\lim_{x \to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$

Example 8: Evaluate $\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2}$

Solution: We can create a common factor by multiplying both numerator and denominator by the conjugate radical expression $\sqrt{x^2+100}+10$ (obtained by changing the sign after the square root). The preliminary algebra rationalizes the numerator:

$$\frac{\sqrt{x^2+100}-10}{x^2} = \frac{\sqrt{x^2+100}-10}{x^2}\cdot\frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}$$

$$= \frac{x^2+100-100}{x^2\left(\sqrt{x^2+100}+10\right)}$$

$$= \frac{x^2}{x^2\left(\sqrt{x^2+100}+10\right)} \qquad \text{Common factor } x^2$$

$$= \frac{1}{\sqrt{x^2+100}+10}. \qquad \text{Cancel } x^2 \text{ for } x \neq 0$$

Therefore,

$$\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2} = \lim_{x\to 0}\frac{1}{\sqrt{x^2+100}+10}$$

$$= \frac{1}{\sqrt{0^2+100}+10} \qquad \text{Denominator not 0 at } x = 0\text{; substitute}$$

$$= \frac{1}{20} = 0.05.$$

2.1.2 Indeterminate Forms:

There are seven indeterminate forms:

$0/0$, ∞/∞, $0.\infty$, $\infty-\infty$, 0^0, ∞^0, and 1^∞

2.1.3 Sandwich Theorem

The following theorem enables us to calculate a variety of limits. It is called the Sandwich Theorem because it refers to a function f whose values are sandwiched between the values of two other functions g and h that have the same limit L at a point c. See Figure 6

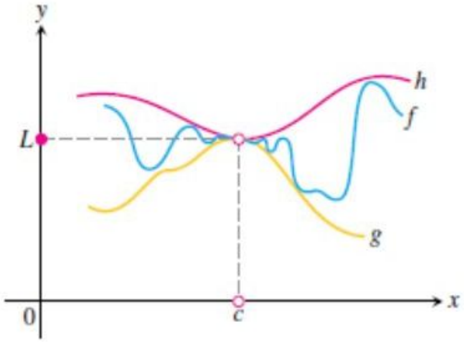


Figure 6

THEOREM 4—The Sandwich Theorem Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

The Sandwich Theorem is also called the Squeeze Theorem or the Pinching Theorem.

Example 9: Given that

$$1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2} \quad \text{for all } x \neq 0$$

find $\lim_{x \to 0} u(x)$, no matter how complicated u is.

Solution: Since

$$\lim_{x \to 0} (1 - (x^2/4)) = 1 \quad \text{and} \lim_{x \to 0} (1 + (x^2/2)) = 1,$$

the Sandwich Theorem implies that $\lim_{x \to 0} u(x) = 1$ (Figure 7).

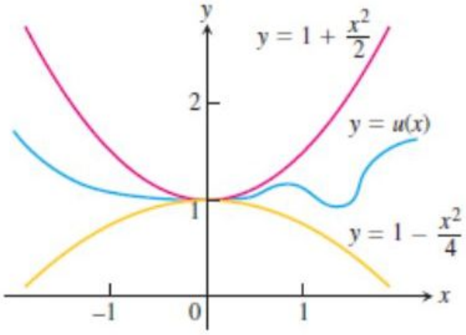


Figure 7

Example 10: Show that $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$.

Solution: First note that we **cannot** use

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = \lim_{x\to 0} x^2 \cdot \lim_{x\to 0} \sin\frac{1}{x}$$

because $\lim_{x\to 0} \sin(1/x)$ does not exist (see Example 4 in Section 2.2). However, since

$$-1 \leqslant \sin\frac{1}{x} \leqslant 1$$

we have, as illustrated by Figure 8,

$$-x^2 \leqslant x^2 \sin\frac{1}{x} \leqslant x^2$$

We know that

$$\lim_{x\to 0} x^2 = 0 \qquad \text{and} \qquad \lim_{x\to 0} (-x^2) = 0$$

Taking $f(x) = -x^2$, $g(x) = x^2 \sin(1/x)$, and $h(x) = x^2$ in the Squeeze Theorem, we obtain

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$$

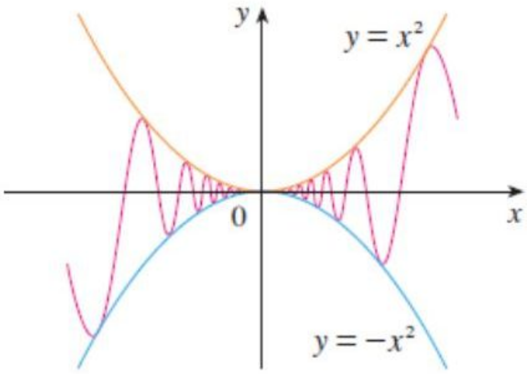
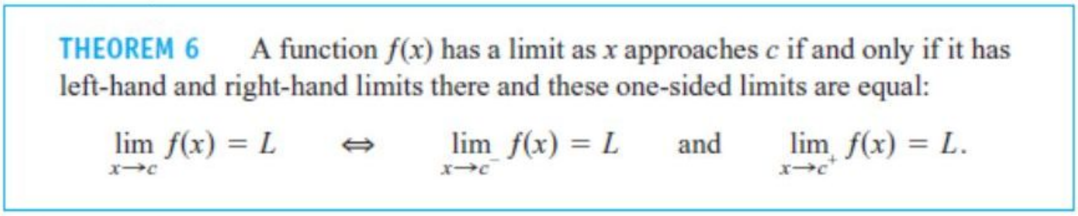


Figure 8

2.1.4 One-Sided Limits

In this section we extend the limit concept to *one-sided limits*, which are limits as *x* approaches the number *c* from the left-hand side (where $x < c$) or the right-hand side ($x > c$) only.



THEOREM 6 A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

In another word:

- Right-hand limit is the limit of *f(x)* as *x* approaches c from the right, or $\lim_{x \to c^+} f(x)$
- Left-hand limit is the limit of *f(x)* as *x* approaches c from the left, or $\lim_{x \to c^-} f(x)$
- $\lim_{x \to c} f(x)$ = L if and only if $\lim_{x \to c^+} f(x) = L$ and $\lim_{x \to c^-} f(x) = L$

Example 11: Let

$$f(x) = \begin{cases} 3 - x, & x < 2 \\ \frac{x}{2} + 1, & x > 2 \end{cases}$$

(a) Find $\lim_{x\to 2^+} f(x)$ and $\lim_{x\to 2^-} f(x)$
(b) Does $\lim_{x\to 2} f(x)$ exist ? why ?

Solution:

(a) $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} f(\frac{x}{2} + 1) = 2/2 + 1 = 2$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(3 - x) = 3 - 2 = 1$

(b) $\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$

$\lim_{x\to 2} f(x)$ does not exist

Example 12: let $f(x)= \begin{cases} 1 - x^2, & x \neq 1 \\ 2, & x = 1 \end{cases}$

(a) Find $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$
(b) Does $\lim_{x\to 1} f(x)$ exist ? why ?
(c) Graph $f(x)$

Solution:

(a) $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 1 - x^2 = 1 - (1)^2 = 0$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 1 - x^2 = 1 - (1)^2 = 0$

(b) $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x)$, $\lim_{x\to 1} f(x)$ exist

2.1.5 Limits Involving (sin θ/θ)

A central fact about (sin θ/θ) is that in radian measure its limit as $\theta \to 0$ is 1.

THEOREM 7

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \qquad (1)$$

Proof The plan is to show that the right-hand and left-hand limits are both 1. Then we will know that the two-sided limit is 1 as well.

To show that the right-hand limit is 1, we begin with positive values of less than π/2 (Figure 9).

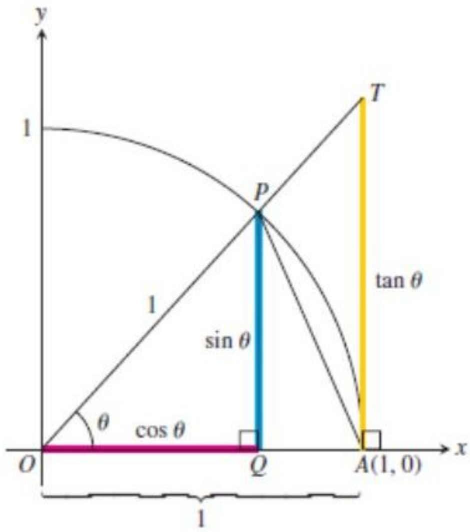


Figure 9

$$\text{Area } \Delta OAP < \text{ area sector } OAP < \text{ area } \Delta OAT.$$

We can express these areas in terms of θ as follows:

$$\begin{aligned} \text{Area } \Delta OAP &= \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\sin\theta) = \frac{1}{2}\sin\theta \\ \text{Area sector } OAP &= \frac{1}{2}r^2\theta = \frac{1}{2}(1)^2\theta = \frac{\theta}{2} \qquad (2) \\ \text{Area } \Delta OAT &= \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\tan\theta) = \frac{1}{2}\tan\theta. \end{aligned}$$

Thus,

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

This last inequality goes the same way if we divide all three terms by the number $(1/2)\sin\theta$, which is positive since $0 < \theta < \pi/2$:

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Taking reciprocals reverses the inequalities:

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Since $\lim_{\theta\to0^+} \cos\theta = 1$, the Sandwich theorem gives:

$$\lim_{\theta\to0^+} \frac{\sin\theta}{\theta} = 1.$$

Recall that $\sin\theta$ and θ are both *odd functions* (Section 1.1). Therefore, $f(\theta) = (\sin\theta)/\theta$ is an *even function*, with a graph symmetric about the y-axis (see Figure 2.32). This symmetry implies that the left-hand limit at 0 exists and has the same value as the right-hand limit:

$$\lim_{\theta\to0^-} \frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to0^+} \frac{\sin\theta}{\theta},$$

so $\lim_{\theta\to0} (\sin\theta)/\theta = 1$ by Theorem 6. ■

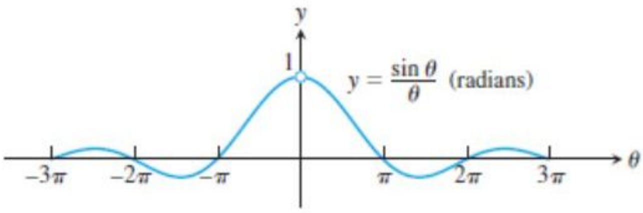


Figure 10

Example 13: show that

(a) $\lim_{h\to0} \frac{\cos h-1}{h} = 0$ and **(b)** $\lim_{x\to0} \frac{\sin 2x}{5x} = \frac{2}{5}$

Solution:

(a) Using the half-angle formula $\cos h = 1 - 2\sin^2(h/2)$, we calculate

$$\lim_{h\to0} \frac{\cos h - 1}{h} = \lim_{h\to0} -\frac{2\sin^2(h/2)}{h}$$

$$= -\lim_{\theta\to0} \frac{\sin\theta}{\theta} \sin\theta \quad \text{Let } \theta = h/2.$$

$$= -(1)(0) = 0. \quad \text{Eq. (1) and Example 11a in Section 2.2}$$

(b) Equation (1) does not apply to the original fraction. We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by $2/5$:

$$\lim_{x\to 0}\frac{\sin 2x}{5x} = \lim_{x\to 0}\frac{(2/5)\cdot\sin 2x}{(2/5)\cdot 5x}$$

$$= \frac{2}{5}\lim_{x\to 0}\frac{\sin 2x}{2x}$$ Now, Eq. (1) applies with $\theta = 2x$.

$$= \frac{2}{5}(1) = \frac{2}{5}$$ ■

Example 14: Find $\lim_{t\to 0}\frac{\tan t\sec 2t}{3t}$

Solution From the definition of $\tan t$ and $\sec 2t$, we have

$$\lim_{t\to 0}\frac{\tan t\sec 2t}{3t} = \frac{1}{3}\lim_{t\to 0}\frac{\sin t}{t}\cdot\frac{1}{\cos t}\cdot\frac{1}{\cos 2t}$$

$$= \frac{1}{3}(1)(1)(1) = \frac{1}{3}.$$ Eq. (1) and Example 11b in Section 2.2 ■

2.1.6 Limits at Infinity

General rules:

$\lim_{x\to\infty}\frac{1}{x} = 0$, $\lim_{x\to-\infty}\frac{1}{x} = 0$

$\lim_{x\to\infty} k = k$, $\lim_{x\to-\infty} k = k$

$\lim_{x\to\infty}\frac{sin\theta}{\theta} = 0$, to prove it:

$-1 \le \sin\theta \le 1$ $[\div\theta]$ $\frac{-1}{\theta} \le \frac{sin\theta}{\theta} \le \frac{1}{\theta}$

$\lim_{\theta\to\infty}\frac{-1}{\theta} = 0$ and $\lim_{\theta\to\infty}\frac{1}{\theta} = 0$, then from Sandwich theorem:

$$\lim_{\theta\to\infty}\frac{sin\theta}{\theta} = 0$$

Limits at Infinity of Rational Functions

There are two methods:

1. Divide both the numerator and the denominator by the highest power of x in denominator.
2. Suppose that $x = 1/h$ and find limit as h approaches zero.

Note: for rational function $\frac{f(x)}{g(x)}$

1. If degree of $f(x)$ less than degree of $g(x)$, then $\lim_{x\to\pm\infty}\frac{f(x)}{g(x)} = 0$
2. If degree of $f(x)$ equals degree of $g(x)$, then $\lim_{x\to\pm\infty}\frac{f(x)}{g(x)}$ is finite.
3. If degree of $f(x)$ greater than degree of $g(x)$, then $\lim_{x\to\pm\infty}\frac{f(x)}{g(x)}$ is infinite.

Example 15: find $\lim_{x\to\infty}\frac{x^3-4x^2+7}{2x^2-3}$

Method 1: $\lim_{x\to\infty}\frac{\frac{x^3}{x^2}-\frac{4x^2}{x^2}+\frac{7}{x^2}}{\frac{2x^2}{x^2}-\frac{3}{x^2}} = \lim_{x\to\infty}\frac{x-4+\frac{7}{x^2}}{2-\frac{3}{x^2}} = \frac{\infty-4+0}{2-0} = \frac{\infty}{2} = \infty$

Method 2: let $x = 1/h$, $h \to 0$

$$\lim_{h\to0}\frac{(\frac{1}{h})^3-4\left(\frac{1}{h}\right)^2+7}{2(\frac{1}{h})^2-3} = \lim_{h\to0}\frac{\frac{1}{h^3}-\frac{4}{h^2}+7}{\frac{2}{h^2}-3} = \lim_{h\to0}\frac{\frac{1-4h+7h^3}{h^3}}{\frac{2-3h^2}{h^2}}$$

$$\lim_{h\to0}\frac{1-4h+7h^3}{h(2-3h^2)} = \frac{1-4(0)+7(0)}{0(2-3(0))} = \frac{1}{0} = \infty$$

Example 16: Find $\lim_{x\to\infty}\frac{x^{3/2}+5}{\sqrt{x^3+4}}$

$$\lim_{x\to\infty}\frac{\frac{x^{3/2}}{x^{3/2}}+\frac{5}{x^{3/2}}}{\sqrt{\frac{x^3}{x^3}+\frac{4}{x^3}}} = \lim_{x\to\infty}\frac{1+\frac{5}{x^{3/2}}}{\sqrt{1+\frac{4}{x^3}}} = \frac{1+0}{\sqrt{1+0}} = \frac{1}{\sqrt{1}} = 1$$

2.1.7 Absolute Value in Limit Problems

Example 17: Find $\lim_{x\to -2}(x+3)\frac{|x+2|}{x+2}$

Solution:

$\lim_{x\to -2^+}(x+3)\frac{|x+2|}{x+2} = \lim_{x\to -2^+}(x+3)\frac{(x+2)}{(x+2)}$

$= \lim_{x\to -2^+}(x+3) = -2 + 3 = 1$

$\lim_{x\to -2^-}(x+3)\frac{|x+2|}{x+2} = \lim_{x\to -2^-}(x+3)\frac{-(x+2)}{(x+2)}$

$= \lim_{x\to -2^-}(x+3)(-1) = (-2+3)(-1) = -1$

2.2 Continuity

We noticed that the limit of a function as x approaches a can often be found simply by calculating the value of the function at a. Functions with this property are called *continuous at a.*

> **[1] DEFINITION** A function f is **continuous at a number a** if
>
> $$\lim_{x\to a} f(x) = f(a)$$

Notice that Definition l implicitly requires three things if f is continuous at a:

1. $f(a)$ is defined (that is, a is in the domain of f)

2. $\lim_{x\to a} f(x)$ exists

3. $\lim_{x\to a} f(x) = f(a)$

Example 18: Where are each of the following functions discontinuous?

(a) $f(x) = \frac{x^2-x-2}{x-2}$

(b) $f(x) = \begin{cases} \frac{1}{x^2} & if\ x \neq 0 \\ 1 & if\ x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & if\ x \neq 2 \\ 1 & if\ x = 2 \end{cases}$

Solution:

(a) Notice that $f(2)$ is not defined, so f is discontinuous at 2. Later we'll see why is continuous at all other numbers.

(b) Here $f(0)$ is defined but

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} 1/x^2$$

does not exist. So f is discontinuous at 0.

(c) Here $f(2)$ is defined and

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-x-2}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+1)}{x-2} = \lim_{x\to 2}(x+1) = 3$$

exists. But

$$\lim_{x \to 2} f(x) \neq f(2)$$

so *f* is not continuous at 2. See Figure 11

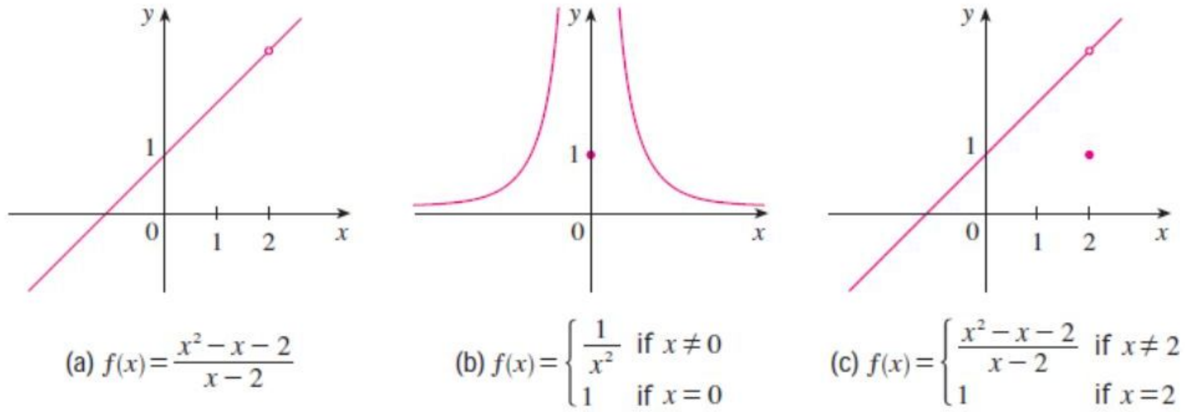


Figure 11

2 DEFINITION A function f is **continuous from the right at a number** a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at** a if

$$\lim_{x \to a^-} f(x) = f(a)$$

3 DEFINITION A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean continuous from the right or continuous from the left.)

$$\begin{cases} 2-x, & 0 \le x \le 1 \\ 2, & 1 \le x \le 2 \\ 3, & x = 2 \\ 2x-2, & 2 < x \le 3 \\ 10-2x, & 3 < x \le 4 \end{cases}$$

Solution:

At $x = 0$:

1. $f(0) = 2 - 0 = 0$
2. $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2 - x = 2 - 0 = 2$
3. $f(0) = \lim_{x\to 0^+} f(x)$

the function is continuous at $x = 0$.

At $x = 1$:

1. $f(1) = 2$
2. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2 = 2$
   $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2 - x = 2 - 1 = 1$
3. $2 \neq 1$ $\lim_{x\to 1} f(x)$ does not exist
   The function is not continuous at $x = 1$

At $x = 2$:

1. $f(2) = 3$
2. $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 2x - 2 = 2\,(2) - 2 = 2$
   $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 2 = 2$
3. $f(2) \neq \lim_{x\to 2}$ the function is not continuous at x = 2

At $x = 3$:

1. $f(3) = 2\,(3) - 2 = 4$
2. $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 10 - 2x = 10 - 2\,(3) = 4$

$\lim_{x\to3^-} f(x) = \lim_{x\to3^-} 2x - 2 = 2(3) - 2 = 4$

$4 = 4 \quad \lim_{x\to3} f(x) = 4$

3. $f(3) = \lim_{x\to3} f(x)$ the function is continuous at $x = 3$

<u>At $x = 4$:</u>

1. $f(4) = 10 - 2(4) = 2$
2. $\lim_{x\to4^-} f(x) = \lim_{x\to4^-} 10 - 2x = 10 - 2(4) = 2$
3. $f(4) = \lim_{x\to4^-} f(x)$ the function is continuous at $x = 4$.

By graphing the function we can check the continuity of the function as shown in Figure

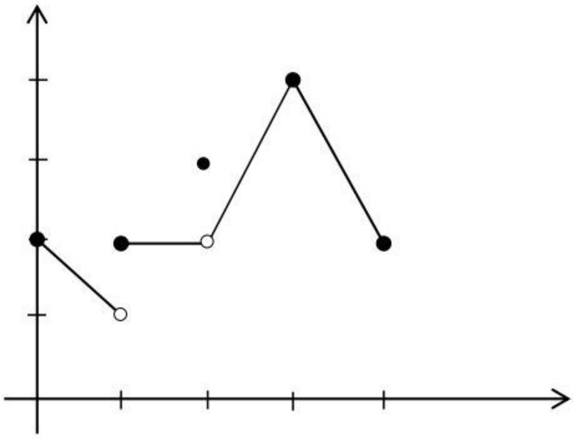


Figure 12

Example 20: At what points is the function $f(x)$ continuous?

$$f(x) = \begin{cases} 1, & x < 0 \\ \sqrt{1 - x^2}, & 0 \le x \le 1 \\ x - 1, & x > 1 \end{cases}$$

Solution:

<u>At $x = 0$</u>

1. $f(0) = \sqrt{1 - (0)^2} = \sqrt{1} = 1$
2. $\lim_{x\to0^+} f(x) = \sqrt{1 - (0)^2} = \sqrt{1} = 1$

   $\lim_{x\to0^-} f(x) = \lim_{x\to0^1} 1 = 1$

$\lim_{x \to 0} f(x) = 1$

3. $f(0) = \lim_{x \to 0} f(x)$ $f(x)$ is continuous at $x = 0$

At $x = 0$

1. $f(1) = 0$
2. $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x - 1 = 1 - 1 = 0$

   $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \sqrt{1 - (1)^2} = 0$

   $\lim_{x \to 1} f(x) = 0$
3. $f(1) = \lim_{x \to 1} f(x)$ $f(x)$ continuous at $x = 1$

For $x < 0$:

$f(x)$ is continuous function

For $x > 1$:

$f(x) = x - 1$ is continuous function

The function is continuous at every point

Example 21: Find the points at which $y = (x^3 - 1) / (x^2 - 1)$ is discontinuous.

Solution:

The function to be discontinuous, the denominator must equal to **zero**:

So that, $x^2 - 1 = 0$ $x^2 = 1$ $x = \pm 1$

Example 22: What value should be assigned to a to make the function

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$

continuous at $x = 3$?

Solution: to make $f(x)$ continuous at $x = 3$:

$\lim_{x\to3} f(x) = f(3)$

$\lim_{x\to3^-} f(x) = \lim_{x\to3^-} x^2 - 1 = (3)^2 - 1 = 8$

$\lim_{x\to3} f(x) = 8$

$f(3) = 2ax = 2 * a * 3 = 6a$

$8 = 6a \quad a = 6/8 = 4/3$

3 DEFINITION A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean continuous from the right or continuous from the left.)

Example 23: Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval [-1, 1].

Solution:
The function does not have a left-hand limit at $x = -1$ or a right-hand limit at $x = 1$.
$\lim_{x\to-1^+} f(x) = 1 = f(-1)$ and
$\lim_{x\to1^-} f(x) = 1 = f(1)$
So f is continuous from the right at -1 and continuous from the left at 1. Therefore, according to Definition, f is continuous on [-1, 1].
The graph of is sketched in Figure . It is the lower half of the circle:

$$x^2 + (y - 1)^2 = 1$$

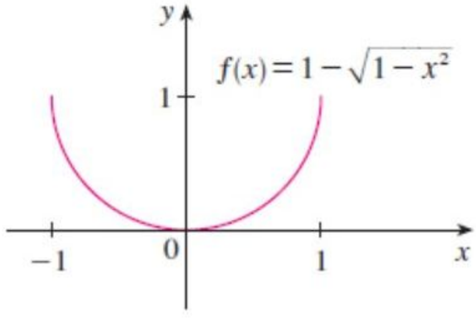


Figure 13

THEOREM 8—Properties of Continuous Functions If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums:* $f + g$
2. *Differences:* $f - g$
3. *Constant multiples:* $k \cdot f$, for any number k
4. *Products:* $f \cdot g$
5. *Quotients:* f/g, provided $g(c) \neq 0$
6. *Powers:* f^n, n a positive integer
7. *Roots:* $\sqrt[n]{f}$, provided it is defined on an open interval containing c, where n is a positive integer

2.2.1 Continuous Extension to a Point

Example 24: Show that

$$f(x) = \frac{x^2+x-6}{x^2-4}, \quad x \neq 2$$

has a continuous extension to $x = 2$, and find that extension.

Solution:

Although $f(2)$ is not defined, if $x \neq 2$ we have

$$f(x) = \frac{x^2+x-6}{x^2-4} = \frac{(x-2)(x+3)}{(x-2)(x+2)} = \frac{x+3}{x+2}$$

The new function

$$F(x) = \frac{x+3}{x+2}$$

is equal to $f(x)$ for $x \neq 2$ but is continuous at $x = 2$, having there the value of 5/4. Thus F is the continuous extension of f to $x = 2$, and:

$$\lim_{x\to 2} \frac{x^2+x-6}{x^2-4} = \lim_{x\to 2} f(x) = \frac{5}{4}$$

$$f(x) = \begin{cases} \dfrac{x^2+x-6}{x^2-4}, & x \neq 2 \\ \dfrac{5}{4}, & x = 2 \end{cases}$$

This form is called the continuous extension of the original function to the $x = 2$.

CHAPTER 3

Differentiation

The problem of finding the tangent line to a curve and the problem of finding the velocity of an object both involve finding the same type of limit, as we saw in chapter two. This special type of limit is called a *derivative* and we will see that it can be interpreted as a rate of change in any of the sciences or engineering.

3.1 Tangents and the Derivative at a Point

To find a tangent to an arbitrary curve $y = f(x)$ at a point $P(x_o, f(x_o))$, we calculate the slope of the secant through P and a nearby point $Q(x_o + h, f(x_o + h))$.We then investigate the limit of the slope as $h \to 0$ (Figure 1). If the limit exists, we call it the slope of the curve at P and define the tangent at P to be the line through P having this slope.

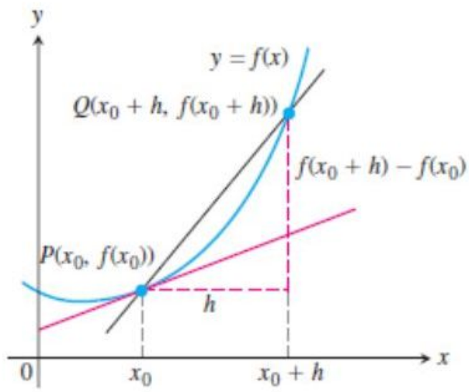


Figure 1

DEFINITIONS The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

Example 1

(a) Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$?

(b) Where does the slope equal – ¼?

(c) What happens to the tangent to the curve at the point (a, 1/a) as a changes?

Solution:

(a) Here $f(x) = 1/x$. The slope at $(a, 1/a)$ is

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\frac{1}{a+h}-\frac{1}{a}}{h} = \lim_{h\to 0} \frac{1}{h}\frac{a-(a+h)}{a(a+h)}$$

$$= \lim_{h\to 0} \frac{-h}{ha(a+h)} = \lim_{h\to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}.$$

Notice how we had to keep writing "$\lim_{h\to 0}$" before each fraction until the stage where we could evaluate the limit by substituting $h = 0$. The number a may be positive or negative, but not 0. When $a = -1$, the slope is $-1/(-1)^2 = -1$ (Figure 2).

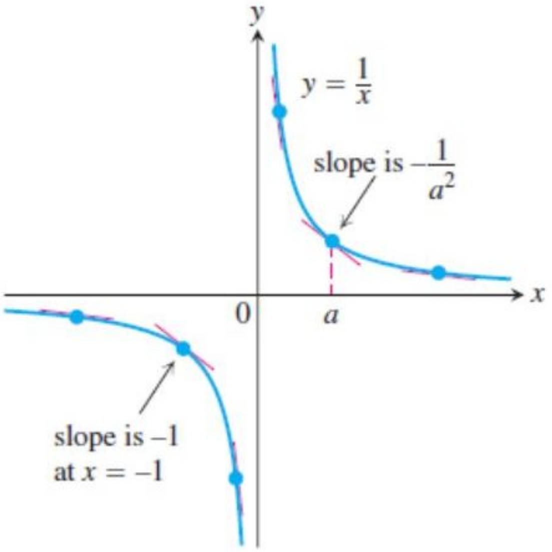


Figure 2

(b) The slope of $y = 1/x$ at the point where $x = a$ is $-1/a^2$. It will be provided that

$$-1/a^2 = -1/4$$

This equation is equivalent to $a^2 = 4$, so $a = 2$ or $a = -2$. The curve has slope -1/4 at the two points (2, 1/2) and (-2, -1/2) (Figure 3).

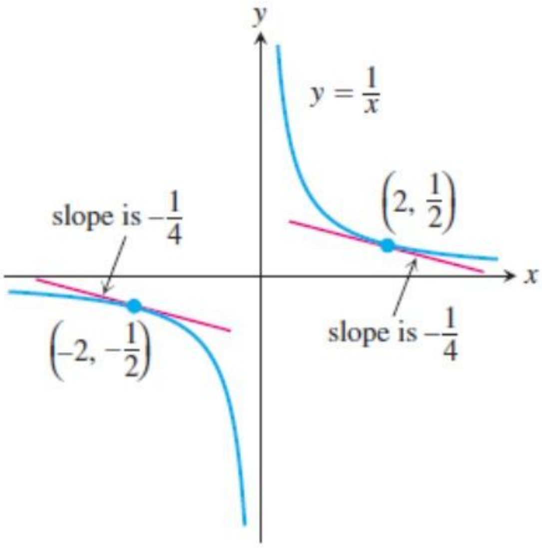


Figure 3

(c) The slope $-1/a^2$ is always negative if $a \neq 0$. As $a \to 0^+$, the slope approaches $-\infty$ and the tangent becomes increasingly steep (Figure 3). We see this situation again as $a \to 0^-$. As a moves away from the origin in either direction, the slope approaches 0 and the tangent levels off to become horizontal.

3.2 Rates of Change: Derivative at a Point

The expression

$$\frac{f(x_0 + h) - f(x_0)}{h}, \quad h \neq 0$$

is called the **difference quotient of f at x_0 with increment h**. If the difference quotient has a limit as h approaches zero, that limit is given a special name and notation.

DEFINITION The **derivative of a function f at a point x_0**, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists.

3.3 Summary

All of these ideas refer to the same limit.

The following are all interpretations for the limit of the difference quotient,

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$
2. The slope of the tangent to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative $f'(x_0)$ at a point

3.4 The Derivative as a Function

In the last section we defined the derivative of $y = f(x)$ at the point $x = x_0$ to be the limit

$$f(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

We now investigate the derivative as a *function* derived from f by considering the limit at each point x in the domain of f.

DEFINITION The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

The process of calculating a derivative is called **differentiation**. To emphasize the idea that differentiation is an operation performed on a function $y = f(x)$, we use the notation

$$\frac{d}{dx} f(x)$$

There are many ways to denote the derivative of a function $y = f(x)$, where the independent variable is x and the dependent variable is y. Some common alternative notations for the derivative are:

$$f(x) = y = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = D(f)(x) = D_x f(x)$$

Example 2: by using the definition of the derivative, find dy/dx of the function $y = 5x^3 + 8x^2 - 3x + 4$

Solution:

$f(x) = 5x^3 + 8x^2 - 3x + 4$

$f(x + \Delta x) = 5(x + \Delta x)^3 + 8(x + \Delta x)^2 - 3(x + \Delta x) + 4$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+h) - f(x)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{5(x + \Delta x)^3 + 8(x + \Delta x)^2 - 3(x + \Delta x) + 4 - 5x^3 - 8x^2 + 3x - 4}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{5(x^3 + 3x^2\,\Delta x + 3x\Delta x^2 + \Delta x^3) + 8(x^2 + 2x\Delta x + \Delta x^2) - 3x - 3\Delta x + 4 - 5x^3 - 8x^2 + 3x - 4}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{5x^3 + 15x^2\,\Delta x + 15x\Delta x^2 + 5\Delta x^3 + 8x^2 + 16x\Delta x + 8\Delta x^2 - 3x - 3\Delta x - 5x^3 - 8x^2 + 3x}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{15x^2\,\Delta x + 15x\Delta x^2 + 5\Delta x^3 + 16x\Delta x + 8\Delta x^2 - 3\Delta x}{\Delta x}$$

$$\lim_{\Delta x \to 0} 15x^2 + 15x\Delta x + 5\Delta x^2 + 16x + 8\Delta x - 3$$

$$= 15\,x^2 + 16\,x - 3$$

3.5 Differentiation Rules

Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Example 3: $\frac{d}{dx}5 = 0$

Power Rule (General Version)

If n is any real number, then

$$\frac{d}{dx}x^n = nx^{n-1},$$

for all x where the powers x^n and x^{n-1} are defined.

Example 4: Differentiate the following equations:

(a) x^3, (b) $x^{2/3}$, (c) $x^{\sqrt{2}}$, (d) $\frac{1}{x^4}$, (e) $x^{-4/3}$, (f) $\sqrt{x^{2+\pi}}$

Solution:

(a) $\frac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$ (b) $\frac{d}{dx}(x^{2/3}) = \frac{2}{3}x^{(2/3)-1} = \frac{2}{3}x^{-1/3}$

(c) $\frac{d}{dx}\left(x^{\sqrt{2}}\right) = \sqrt{2}x^{\sqrt{2}-1}$ (d) $\frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$

(e) $\frac{d}{dx}(x^{-4/3}) = -\frac{4}{3}x^{-(4/3)-1} = -\frac{4}{3}x^{-7/3}$

(f) $\frac{d}{dx}\left(\sqrt{x^{2+\pi}}\right) = \frac{d}{dx}\left(x^{1+(\pi/2)}\right) = \left(1 + \frac{\pi}{2}\right)x^{1+(\pi/2)-1} = \frac{1}{2}(2 + \pi)\sqrt{x^{\pi}}$ ■

Derivative Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

Example 5: Differentiate the equation $y = 3x^2$

Solution: $\frac{dy}{dx}(3x^2) = 3 * 2x = 6x$

Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Example 6: Find the derivative of the polynomial $y = x^3 + (4/3)x^2 - 5x + 1$.

Solution: $\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$

$= 3x^2 + (4/3)*2x - 5 + 0$

$= 3x^2 + (8/3)*2x - 5$

Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Example 7: find the derivative of $y = (x^2 + 1)\,(x^3 + 3)$.

Solution:

(a) From the Product Rule we find:

$$\frac{d}{dx}\left[(x^2+1)(x^3+3)\right] = (x^2+1)(3x^2) + (x^3+3)(2x) \qquad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= 3x^4 + 3x^2 + 2x^4 + 6x$$

$$= 5x^4 + 3x^2 + 6x.$$

(b) This particular product can be differentiated as well (perhaps better) by multiplying out the original expression for y and differentiating the resulting polynomial:

$$y = (x^2+1)(x^3+3) = x^5 + x^3 + 3x^2 + 3$$

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x.$$

Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Example 8: find the derivative of $y = \frac{t^2-1}{t^{3+1}}$

Solution: apply the Quotient Rule:

$$\frac{dy}{dt} = \frac{(t^3+1)\cdot 2t - (t^2-1)\cdot 3t^2}{(t^3+1)^2} \qquad \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2}$$

$$= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3+1)^2}$$

$$= \frac{-t^4 + 3t^2 + 2t}{(t^3+1)^2}.$$

Example 9: Find an equation for the tangent to the curve $y = x + 1/x$ at x = 2.

Solution:

At $x = 2$:

$y = 2 + ½ = 5/2$

point (2, 5/2)

$y = x + \frac{1}{x}$

$\frac{dy}{dx} = 1 + \frac{(x)(0)-(1)(1)}{x^2}$

$= 1 + \frac{-1}{x^2}$

At $x = 2$:

$\frac{dy}{dx} = m = 1 - \frac{1}{4} = ¾$

$(y - y_1) = m\ (x - x_1)$

$(y - 5/2) = 3/4\ (x - 2)$

$(2y - 5)/2 = 3(x - 2)/4$

$8y - 20 = 6(x - 2)$

$8y - 20 = 6x - 12$

$8y - 6x - 8 = 0$

$y = 6/8\ x + 1$

Example 10: Find the point on the curve $y = x^3 + x^2 - 1$ where the tangent is parallel to the x- axis.

Solution:

Slope $= dy/dx = 3x^2 + 2x$

When the tangent is parallel to the x-axis, $m = 0$.

$3x^2 + 2x = 0$

$x\ (3x + 2) = 0$

$x = 0$ or $3x + 2 = 0$ $x = -2/3$

at $x = 0$ $y = -1$

P_1 (0, -1)
At $x = -2/3 \quad y = -23/27$
$P_2 = (-2/3, -23/27)$

Example 11: Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?
Solution: The horizontal tangents, if any, occur where the slope dy/dx is zero. We have:

$$dy/dx = d/dx\,(x^4 - 2x^2 + 2)$$
$$= 4x^3 - 4x$$

Now solve the equation $dy/dx = 0$ for x:

$$4x^3 - 4x = 0$$
$$4x(x^2 - 1) = 0$$
$$x = 0,\ 1,\ -1$$

The curve $y = x4 - 2x2 + 2$ has horizontal tangents at $x = 0$, 1 and -1. The corresponding points on the curve are (0, 2), (1, 1) and (-1, 1)

3.6 Derivatives of Trigonometric Functions

3.6.1 Derivative of the Sine Function

$$\sin(x + h) = \sin x \cos h + \cos x \sin h.$$

If $f(x) = \sin x$, then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \qquad \text{Derivative definition}$$

$$= \lim_{h\to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} = \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h\to 0} \left(\sin x \cdot \frac{\cos h - 1}{h}\right) + \lim_{h\to 0} \left(\cos x \cdot \frac{\sin h}{h}\right)$$

$$= \sin x \cdot \underbrace{\lim_{h\to 0} \frac{\cos h - 1}{h}}_{\text{limit } 0} + \cos x \cdot \underbrace{\lim_{h\to 0} \frac{\sin h}{h}}_{\text{limit } 1} = \sin x \cdot 0 + \cos x \cdot 1 = \cos x.$$

Example 5a and Theorem 7, Section 2.4

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x.$$

Example 12: find the derivatives of the following functions:

(a) $y = \sin x \cos x$

(b) $y = \frac{cosx}{1-sinx}$

Solution:

(a)

$y = \sin x \cos x$:

$$\begin{aligned}\frac{dy}{dx} &= \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x) \\ &= \sin x(-\sin x) + \cos x(\cos x) \\ &= \cos^2 x - \sin^2 x\end{aligned}$$

(b)

$y = \frac{\cos x}{1 - \sin x}$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1 - \sin x)\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} \\ &= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2} \\ &= \frac{1 - \sin x}{(1 - \sin x)^2} \\ &= \frac{1}{1 - \sin x}\end{aligned}$$

3.6.2 Derivatives of the Other Basic Trigonometric Functions

The derivatives of the other trigonometric functions:

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Example 13: derive the following equations:

$y = \cos x \tan 3x$

$y = \tan\sqrt{3x}$

$y = sin^2(\frac{1}{x})$

Solution:

$dy/dx = \cos x\,(\sec^2 3x * 3) + \tan 3x\,(-\sin x)$

$\quad = 3\cos x \sec^2 3x - \sin x \tan 3x$

$dy/dx = \sec^2 (3x)^{1/2} * ½\,(3x)^{-1/2} * 3$

$\quad = \frac{3}{2\sqrt{3x}}\, sec^2\sqrt{3x}$

$y = (\sin\left(\frac{1}{x}\right))^2$

$dy/dx = 2\,(\sin\left(\frac{1}{x}\right) \times \cos\left(\frac{1}{x}\right) \times (-1 \times x^{-2}))$

$dy/dx = 2(\sin\left(\frac{1}{x}\right) \times \cos\left(\frac{1}{x}\right) \times (\frac{-1}{x^2})$

$\quad = \frac{-2}{x^2}\sin\frac{1}{x}\cos\frac{1}{x}$

Example 14: find the point on the curve $y = \tan x$, $-\pi/2 < x < \pi/2$, where the tangent is parallel to the line $y = 2x$

Solution:

Slope of the line $y = 2x$ is $dy/dx = 2$

Slope of the curve $y = \tan x$ should be equal to 2 (parallel to line $y = 2x$)

$dy/dx = \sec^2 x = \frac{1}{cos^2 x}$

$$\frac{1}{cos^2 x} = 2$$

$\cos^2 x = ½$

$\cos x = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$

If $\cos x = -\frac{1}{\sqrt{2}}$ x out of interval $(-\pi/2, \pi/2)$

If $\cos x = \frac{1}{\sqrt{2}}$ $x = \pi/4$ and $x = -\pi/4$

For $x = \pi/4$ $y = \tan \pi/4 = 1$

For $x = -\pi/4$ $y = \tan -\pi/4 = -1$

The points are $(\pi/4, 1)$ and $(-\pi/4, -1)$

3.7 The Chain Rule

The derivative of the composite function $f(g(x))$ at x is the derivative of f at $g(x)$ times the derivative of g at x. This is known as the Chain Rule (Figure 4).

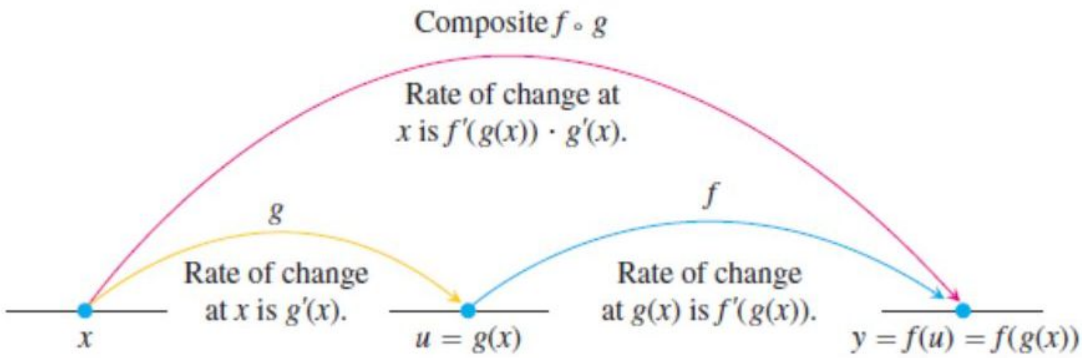


Figure 4

THEOREM 2—The Chain Rule If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

Example 15: If $y = x^2 + 2x + 1$, $x = 3u^2 + 1$, find dy/du.

Solution:
Method 1: substitute x function in y function:
$y = (3u^2 + 1)^2 + 2(3u^2 + 1) + 1$
$= 9u^4 + 6u^2 + 1 + 6u^2 + 2 + 1$
$= 9u^4 + 12u^2 + 4$
$dy/dx = 36u^3 + 24u$

Method 2: Chain Rule
$dy/du = dy/dx * dx/du$
$dy/dx = 2x + 2, \quad dx/du = 6u$
$dy/du = (2x + 2)\ (6u) = [\ 2(3u^2 + 1) + 2]\ (6u)$
$= (6u^2 + 4)\ (6u)$
$= (36u^3 - 24u)$

Example 16: Find dy/dt for $y = \sin\ (t^2 + 6)$ by using Chain Rule
Solution:
Let $y = \sin u$ and $u = t^2 + 6$
$dy/dt = dy/du * du/dt$
$dy/du = \cos u, \quad du/dt = 2t$
$dy/dx = \cos u * 2t$
$= \cos\ (t^2 + 6) * 2t$
$= 2t \cos(t^2 + 6)$

3.8 Repeated Use of the Chain Rule

We sometimes have to use the Chain Rule two or more times to find a derivative.

Example 17: Find the derivative of function $g(t) = \tan(5 - \sin 2t)$.
Solution:
Notice here that the tangent is a function of $5 - \sin 2t$ whereas the sine is a function of $2t$, which is itself a function of t. Therefore, by the Chain Rule:

$$\begin{aligned}
g'(t) &= \frac{d}{dt}(\tan(5 - \sin 2t)) \\
&= \sec^2(5 - \sin 2t) \cdot \frac{d}{dt}(5 - \sin 2t) \\
&= \sec^2(5 - \sin 2t) \cdot \left(0 - \cos 2t \cdot \frac{d}{dt}(2t)\right) \\
&= \sec^2(5 - \sin 2t) \cdot (-\cos 2t) \cdot 2 \\
&= -2(\cos 2t)\sec^2(5 - \sin 2t).
\end{aligned}$$

Example 18: Show that the slope of every line tangent to the curve $y = 1/(1 - 2x)^3$ is positive.
Solution We find the derivative:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(1 - 2x)^{-3} \\
&= -3(1 - 2x)^{-4} \cdot \frac{d}{dx}(1 - 2x) \\
&= -3(1 - 2x)^{-4} \cdot (-2) \\
&= \frac{6}{(1 - 2x)^4}.
\end{aligned}$$

At any point (x, y) on the curve, $x \neq ½$ and the slope of the tangent line is :

$$\frac{dy}{dx} = \frac{6}{(1 - 2x)^4}$$

the quotient of two positive numbers.

3.9 Implicit Differentiation

Most of the functions we have dealt with so far have been described by an equation of the form $y = f(x)$ that expresses y explicitly in terms of the variable x. We have learned rules for differentiating functions defined in this way. Another situation occurs when we encounter equations like

$$x^3 + y^3 - 9xy = 0,\ y^2 - x = 0 \text{ or } x^2 + y^2 - 25 = 0.$$

These equations define an **implicit** relation between the variables x and y. In some cases we may be able to solve such an equation for y as an explicit function (or even several functions) of x. When we cannot put an equation $F(x, y) = 0$ in the form $y = f(x)$ to differentiate it in the usual way, we may still be able to find dy/dx by **implicit differentiation**. This section describes the technique.

> **Implicit Differentiation**
>
> 1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
> 2. Collect the terms with dy/dx on one side of the equation and solve for dy/dx.

Example 19:

(a) If $x^2 + y^2 = 25$, find dy/dx.

(b) Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point (3, 4).

Solution:

(a) Differentiate both sides of the equation $x^2 + y^2 = 25$

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

Remembering that y is a function of x and using the Chain Rule, we have

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx} = 2y\frac{dy}{dx}$$

Thus

$$2x + 2y\frac{dy}{dx} = 0$$

Now we solve this equation for *dy/dx:*

$$dy/dx = -x/y$$

(b) At the point (3, 4) we have $x = 3$ and $y = 4$, so

$$dy/dx = -3/4$$

An equation of the tangent to the circle at (3, 4) is therefore

$$y - 4 = -3/4\ (x - 3) \quad \text{or } 3x + 4y = 25$$

Example 20: Find *dy/dx* if $y^2 = x^2 + \sin xy$.
Solution We differentiate the equation implicitly.

$$y^2 = x^2 + \sin xy$$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin xy)$$ Differentiate both sides with respect to x ...

$$2y\frac{dy}{dx} = 2x + (\cos xy)\frac{d}{dx}(xy)$$ ... treating y as a function of x and using the Chain Rule.

$$2y\frac{dy}{dx} = 2x + (\cos xy)\left(y + x\frac{dy}{dx}\right)$$ Treat xy as a product.

$$2y\frac{dy}{dx} - (\cos xy)\left(x\frac{dy}{dx}\right) = 2x + (\cos xy)y$$ Collect terms with dy/dx.

$$(2y - x\cos xy)\frac{dy}{dx} = 2x + y\cos xy$$

$$\frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}$$ Solve for dy/dx.

3.10 Derivatives of Higher Order

Example21: Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

Solution:
To start, we differentiate both sides of the equation with respect to x in order to find $y = dy/dx$.

$$\frac{d}{dx}(2x^3 - 3y^2) = \frac{d}{dx}(8)$$

$$6x^2 - 6yy' = 0 \qquad \text{Treat } y \text{ as a function of } x.$$

$$y' = \frac{x^2}{y}, \quad \text{when } y \neq 0 \qquad \text{Solve for } y'.$$

We now apply the Quotient Rule to find y''.

$$y'' = \frac{d}{dx}\left(\frac{x^2}{y}\right) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2} \cdot y'$$

Finally, we substitute $y' = x^2/y$ to express y'' in terms of x and y.

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2}\left(\frac{x^2}{y}\right) = \frac{2x}{y} - \frac{x^4}{y^3}, \quad \text{when } y \neq 0$$

■

Example 22: Show that the point (2, 4) lies on the curve $x^3 + y^3 - 9xy = 0$. Then find the tangent and normal to the curve there (Figure 5).

Solution: The point (2, 4) lies on the curve because its coordinates satisfy the equation given for the curve:

$$2^3 + 4^3 - 9\,(2)\,(4) = 8 + 64 - 72 = 0$$

To find the slope of the curve at (2, 4), we first use implicit differentiation to find a formula for dy/dx:

$$x^3 + y^3 - 9xy = 0$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - \frac{d}{dx}(9xy) = \frac{d}{dx}(0)$$

$$3x^2 + 3y^2\frac{dy}{dx} - 9\left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) = 0 \qquad \text{Differentiate both sides with respect to } x.$$

$$(3y^2 - 9x)\frac{dy}{dx} + 3x^2 - 9y = 0 \qquad \text{Treat } xy \text{ as a product and } y \text{ as a function of } x.$$

$$3(y^2 - 3x)\frac{dy}{dx} = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x}. \qquad \text{Solve for } dy/dx.$$

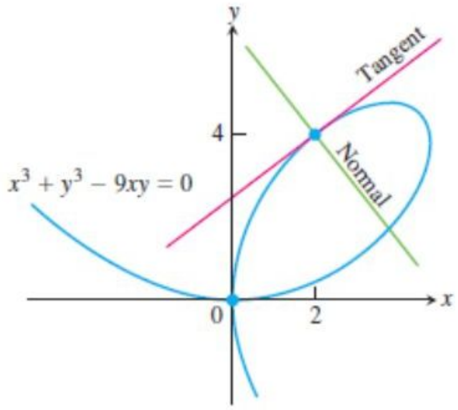


Figure 5

We then evaluate the derivative at $(x, y) = (2, 4)$:

$$\left.\frac{dy}{dx}\right|_{(2,4)} = \left.\frac{3y - x^2}{y^2 - 3x}\right|_{(2,4)} = \frac{3(4) - 2^2}{4^2 - 3(2)} = \frac{8}{10} = \frac{4}{5}.$$

The tangent at (2, 4) is the line through (2, 4) with slope 4/5:

$$y = 4 + \frac{4}{5}(x - 2)$$

$$y = \frac{4}{5}x + \frac{12}{5}.$$

The normal to the curve at (2, 4) is the line perpendicular to the tangent there, the line through (2, 4) with slope – 5/4:

$$y = 4 - \frac{5}{4}(x - 2)$$

$$y = -\frac{5}{4}x + \frac{13}{2}.$$

3.11 Parametric Equations

If $x = f(t)$ and $y = g(t)$, then these equations are called parametric equations and the variable t is called parameter.

From Chain Rule: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ $\frac{dy}{dx} = \frac{\frac{dy}{du}}{\frac{dx}{du}}$

$x = f(t)$, $y = g(t)$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ the 1st derivative for parametric equation

For second derivative:

$\frac{d^2y}{dx^2} = \frac{dy/dt}{dx/dt}$, $y = \frac{dy}{dx}$

Or $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{dx/dt}$ the 2nd derivative for parametric equation

Example 23: if $y = 2t^3 + 3$, $x = t/(t-1)$, find dy/dx

Solution:

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$dy/dt = 6\,t^2$

$dx/dt = \frac{(t-1)(1)-t(1)}{(t-1)^2} = \frac{-1}{(t-1)^2}$

$dy/dx = \frac{6t^2}{\frac{-1}{(t-1)^2}}$

$dy/dx = -6t^2\,(t-1)^2$

Example 24: If a point traces the circle $x^2 + y^2 = 25$ and if $dx/dt = 4$ when the point reaches (3, 4). Find dy/dt

Solution:

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$x^2 + y^2 = 25 \quad 2x + 2y\,(dy/dx) = 0 \qquad dy/dx = -\,x/y$

At point (3, 4) $dy/dx = -\,3/4$

$-3/4 = \frac{dy/dt}{4}$

$dy/dt = -\,3$

Example 25: If $x = \cos 3t$, $y = \sin^2 3t$, find dy/dx, d^2y/dx^2

Solution:

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$dy/dt = 2\sin 3t\,(\cos 3t).3 = 6\sin 3t\cos 3t$

$dx/dt = -\sin 3t\,.3 = -3\sin 3t$

$\frac{dy}{dx} = \frac{6\sin 3t\cos 3t}{-3\sin 3t} = -2\cos 3t = -\,2x$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{dx/dt}$$

$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}(-2\cos 3t)$= *-2 (- sin 3t).3*

$\frac{d^2y}{dx^2} = \frac{-2(-\sin 3t).3}{-3\sin 3t}$ = -2

Or $dy/dx = - 2x$

$d^2y/dx^2 = -2$

when *dy/dx* with respect to *x*.

CHAPTER 4

Applications of Derivatives

4.1 Related Rates

In this section we look at problems that ask for the rate at which some variable changes when it is known how the rate of some other related variable (or perhaps several variables) changes. The problem of finding a rate of change from other known rates of change is called a *related rates problem.*

Example 1: Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

Solution: Figure 1 shows a partially filled conical tank. The variables in the problem are:

V = volume (ft^3) of the water in the tank at time t (min)
x = radius (ft) of the surface of the water at time t
y = depth (ft) of the water in the tank at time t.

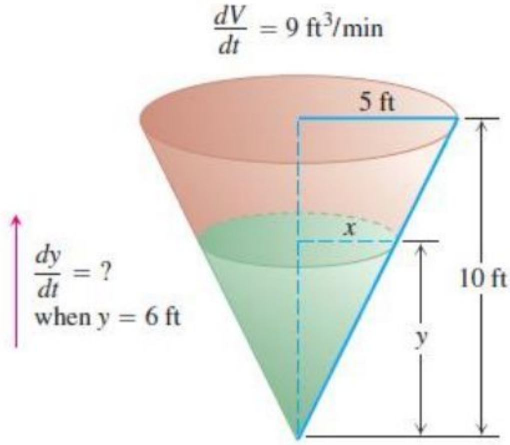


Figure 1

We assume that *V, x*, and *y* are differentiable functions of *t*. The constants are the dimensions of the tank. We are asked for *dy/dt* when

$$y = 6 \text{ ft} \quad \text{and } dV/dt = 9 \text{ ft}^3/\text{min}.$$

The water forms a cone with volume

$$V = \frac{1}{3}\pi x^2 y.$$

This equation involves *x* as well as *V* and *y*. Because no information is given about *x* and *dx/dt* at the time in question, we need to eliminate *x*. The similar triangles in Figure 1 give us a way to express *x* in terms of *y*:

$$\frac{x}{y} = \frac{5}{10} \text{ or } x = \frac{y}{2}$$

Therefore, find

$$V = \frac{1}{3}\pi(\frac{y}{2})^2 y = \frac{\pi}{12}y^3$$

To give the derivative

$$\frac{dV}{dt} = \frac{\pi}{12}.3y^2\frac{dy}{dt} = \frac{\pi}{4}y^2\frac{dy}{dt}$$

Finally, use $y = 6$ and $dV/dt = 9$ to solve for *dy/dt*.

$$9 = \frac{\pi}{4}(6)^2\frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{1}{\pi} \approx 0.32$$

At the moment in question, the water level is rising at about 0.32 ft /min.

Related Rates Problem Strategy

1. *Draw a picture and name the variables and constants*. Use *t* for time. Assume that all variables are differentiable functions of *t*.
2. *Write down the numerical information* (in terms of the symbols you have chosen).
3. *Write down what you are asked to find* (usually a rate, expressed as a derivative).

4. *Write an equation that relates the variables.* You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.
5. *Differentiate with respect to t.* Then express the rate you want in terms of the rates and variables whose values you know.
6. *Evaluate.* Use known values to find the unknown rate.

Example 2: A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the liftoff point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

Solution We answer the question in six steps:

1. *Draw a picture and name the variables and constants* (Figure 2). The variables in the picture are:

θ = the angle in radians the range finder makes with the ground.

y = the height in feet of the balloon.

We let t represent time in minutes and assume that θ and y are differentiable functions of t.

The one constant in the picture is the distance from the range finder to the liftoff point (500 ft). There is no need to give it a special symbol.

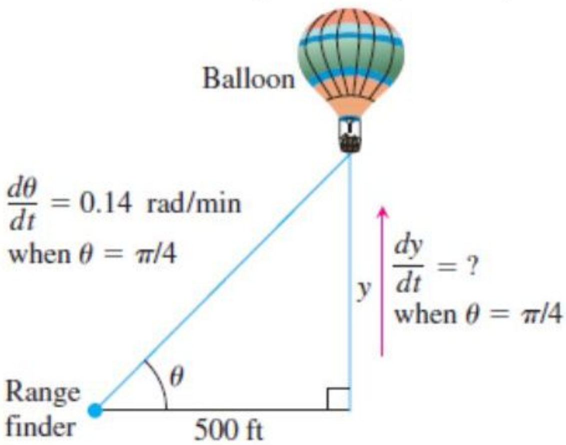


Figure 2

2. *Write down the additional numerical information.*

$$d\theta/dt = 0.14 \text{ rad/min} \quad \text{when } \theta = \pi/4$$

3. *Write down what we are to find.* We want dy/dt when $\theta = \pi/4$
4. *Write an equation that relates the variables y and* θ

$$y/500 = \tan\theta \quad \text{or } y = 500 \tan\theta$$

5. *Differentiate with respect to t using the Chain Rule.* The result tells how (which we want) is related to (which we know).

$$dy/dt = 500\,(\sec^2\theta)\; d\theta/dt$$

6. *Evaluate with* $\theta = \pi/4$ *and* $d\theta/dt = 0.14$ *to find* dy/dt.

$$dy/dt = 500\,(\sqrt{2})^2(0.14) = 140 \quad [\sec \pi/4 = \sqrt{2}]$$

At the moment in question, the balloon is rising at the rate of 140 ft/min.

Example 3: A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

Solution: We picture the car and cruiser in the coordinate plane, using the positive x-axis as the eastbound highway and the positive y-axis as the southbound highway (Figure 3).
We let t represent time and set

x = position of car at time t
y = position of cruiser at time t
s = distance between car and cruiser at time t .

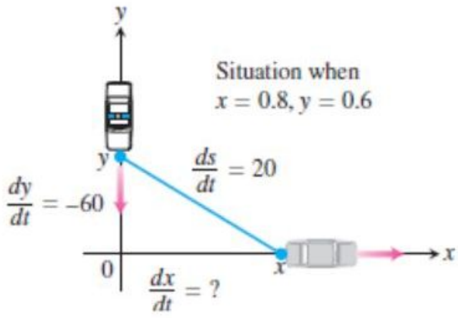


Figure 3

We assume that x, y, and s are differentiable functions of t.

We want to find dx/dt when

$x = 0.8$ mi, $y = 0.6$ mi, $dy/dt = -60$ mph, $ds/dt = 20$ mph.

Note that dy/dt is negative because y is decreasing.
We differentiate the distance equation

$$s^2 = x^2 + y^2$$

(we could also use $s = \sqrt{x^2 + y^2}$), and obtain

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$$= \frac{1}{\sqrt{x^2 + y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right).$$

Finally, we use $x = 0.8$, $y = 0.6$, $dy/dt = -60$, $ds/dt = 20$, and solve for dx/dt.

$$20 = \frac{1}{\sqrt{(0.8)^2 + (0.6)^2}}\left(0.8\frac{dx}{dt} + (0.6)(-60)\right)$$

$$\frac{dx}{dt} = \frac{20\sqrt{(0.8)^2 + (0.6)^2} + (0.6)(60)}{0.8} = 70$$

At the moment in question, the car's speed is 70 mph.

Example 4: A water trough is 10 m long and a cross section has the shape of isosceles trapezoid as shown in Figure 4. If the trough is being filled with water at the rate of 0.2 m^3/ min, how fast is the water level rising when the water is 30 cm deep?

Solution:

V = volume of water

$V = \frac{(0.3+2x)+0.3}{2} * h * 10 = (0.6 + 2x) * 5h = 3h + 10xh$

From similarity of triangles (Figure 4):

$\frac{x}{h} = \frac{0.25}{0.5} = \frac{1}{2}$

$x = h/2$

$V = 3\frac{dh}{dt} + 10\, h\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{dv/dt}{3+10h}$

At $h = 30\ cm = 0.3\ m$ $\quad \frac{dh}{dt} = \frac{0.2}{3+10(0.3)} = \frac{0.2}{6} = \frac{1}{30}\ m/min$

Example 5: A jet airliner is flying at a constant altitude of 12,000 ft above sea level as it approaches a Pacific island. The aircraft comes within the direct line of sight of a radar station located on the island, and the radar indicates the initial angle between sea level and its line of sight to the aircraft is 30°. How fast (in miles per hour) is the aircraft approaching the island when first detected by the radar instrument if it is turning upward (counterclockwise) at the rate of in order to keep the aircraft within its direct line of sight?

Solution: The aircraft *A* and radar station *R* are pictured in the coordinate plane, using the positive x-axis as the horizontal distance at sea level from *R* to *A*, and the positive y-axis as the vertical altitude above sea level. We let t represent time and observe that $y = 12000$ is a constant. The general situation and line-of-sight angle θ are depicted in Figure 5. We want to find dx/dt when $\theta = \pi/6$ rad and $d\theta/dt = 2/3$ deg/sec.

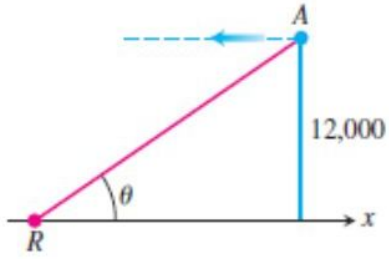


Figure 5

From Figure 5, we see that

$$12{,}000/\,x = \tan\theta \quad \text{or } x = 12{,}000 \cot\theta$$

Using miles instead of feet for our distance units, the last equation translates to

$$x = \frac{12{,}000}{5280}\cot\theta.$$

Differentiation with respect to t gives

$$\frac{dx}{dt} = -\frac{12{,}00}{528}\csc^2\theta\frac{d\theta}{dt}$$

When $\theta = \pi/6$, $\sin^2\theta = 1/4$,so $\csc^2\theta = 4$. Converting $d\theta/dt = 2/3$ *deg/sec* to radians per hour, we find

$$\frac{d\theta}{dt} = \frac{2}{3}\left(\frac{\pi}{180}\right)(3600) \text{ rad/hr} \qquad [\text{ 1 hr} = 3600 \text{ sec, } 1 \text{ deg} = \pi/180 \text{ rad}]$$

Substitution into the equation for dx/dt then gives

$$\frac{dx}{dt} = \left(-\frac{1200}{528}\right)(4)\left(\frac{2}{3}\right)\left(\frac{\pi}{180}\right)(3600) \approx -380$$

The negative sign appears because the distance x is decreasing, so the aircraft is approaching the island at a speed of approximately 380 mi/hr when first detected by the radar.

Example 6: Figure 6 (a) shows a rope running through a pulley at P and bearing a weight W at one end. The other end is held 5 ft above the ground in the hand M of a worker. Suppose the pulley is 25 ft above ground, the rope is 45 ft long, and the worker is walking rapidly away from the vertical line *PW* at the rate of 6 ft/sec. How fast is the weight being raised when the worker's hand is 21 ft away from *PW*?

Solution: We let *OM* be the horizontal line of length x ft from a point *O* directly below the pulley to the worker's hand *M* at any instant of time (Figure 6). Let h be the height of the weight *W* above *O*, and let z denote the length of rope from the pulley *P* to the worker's hand. We want to know dh/dt when $x = 21$ given that $dx/dt = 6$.

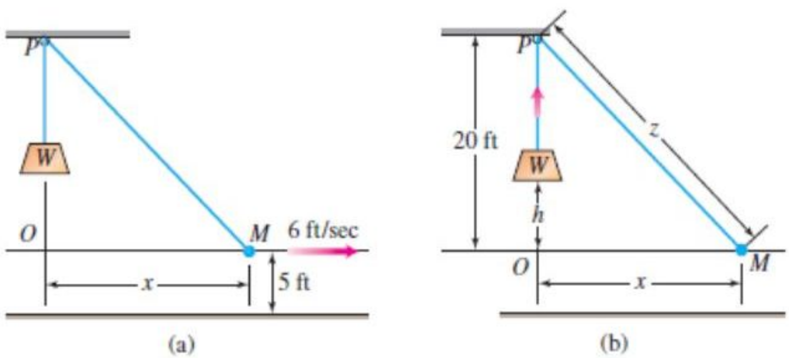


Figure 6

Note that the height of *P* above *O* is 20 ft because *O* is 5 ft above the ground. We assume the angle at *O* is a right angle.

At any instant of time t we have the following relationships (see Figure 5b):

$$20 - h + z = 45 \quad \text{[Total length of rope is 45 ft]}$$

$$20^2 + x^2 = z^2 \text{ [Angle at } O \text{ is a right angle]}$$

If we solve for $z = 25 + h$ in the first equation, and substitute into the second equation, we have

$$20^2 + x^2 = (25 + h)^2 \quad \ldots\ldots (1)$$

Differentiating both sides with respect to t gives

$$2x\frac{dx}{dt} = 2(25 + h)\frac{dh}{dt}$$

and solving this last equation for dh/dt we find

$$\frac{dh}{dt} = (\frac{x}{25+h})\frac{dx}{dt} \quad \ldots\ldots. (2)$$

Since we know dx/dt, it remains only to find $25 + h$ at the instant when $x = 21$. From Equation (1),

$$20^2 + 21^2 = (25 + h)^2$$

So that

$$(25 + h)^2 = 841 \quad \text{or} \quad 25 + h = 29$$

Equation (2) now gives

$$dh/dt = (21/29)*6 \approx 4.3 \text{ ft/sec}$$

as the rate at which the weight is being raised when $x = 21$ ft

4.2 Extreme Values of Functions

This section shows how to locate and identify extreme (maximum or minimum) values of a function from its derivative. Once we can do this, we can solve a variety of problems in which we find the optimal (best) way to do something in a given situation. Finding maximum and minimum values is one of the most important applications of the derivative.

Definitions: Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

Note: Functions with the same defining rule or formula can have different extrema (maximum or minimum values), depending on the domain.
For example, on the closed interval $[-\pi/2, \pi/2]$ the function $f(x) = \cos x$ takes on an absolute maximum value of 1 (once) and an absolute minimum value of 0 (twice). On the same interval, the function $g(x) = \sin x$ takes on a maximum value of 1 and a minimum value of -1 (Figure 7).

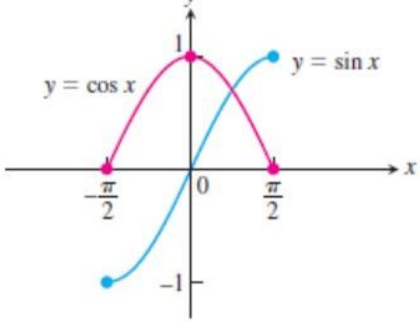


Figure 7

The Extreme Value Theorem: If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$ and $m \leq f(x) \leq M$ for every other x in $[a, b]$.

4.2.1 The Mean Value Theorem

Rolle's Theorem Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f(c) = 0$. (Figure 8)

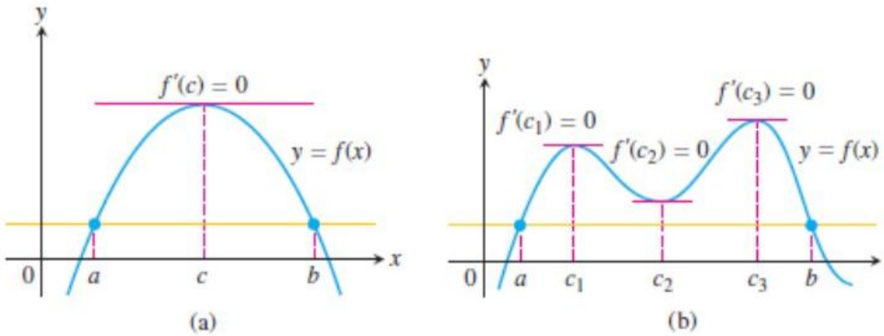


Figure 8

4.2.2 The Mean Value Theorem

Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b) (Figure 9). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f(c)$$

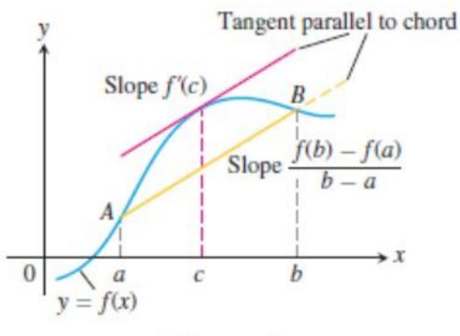


Figure 9

Example 7: if $f(x) = x^2$, $0 \le x \le 2$. Find c by using the mean value theorem.

Solution: $f(x) = 2x$, $f(0) = 0$, $f(2) = 4$

$$f(c) = \frac{f(b) - f(a)}{b - a} = \frac{4 - 0}{2 - 0} = 2$$

$f(c) = 2c$

$2c = 2$

$c = 1$

See Figure 10

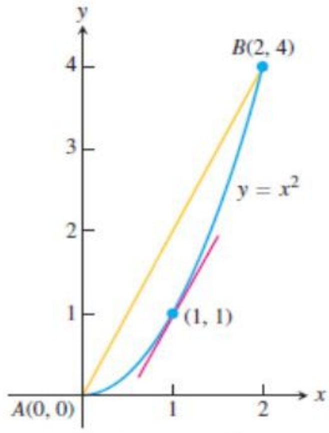


Figure 10

Example 8: To illustrate the Mean Value Theorem with a specific function, let's consider $f(x) = x^3 - x$, $a = 0$, $b = 2$.

Solution:

Since f is a polynomial, it is continuous and differentiable for all x, so it is certainly continuous on [0, 2] and differentiable on (0, 2).

Therefore, by the Mean Value Theorem, there is a number c in (0, 2) such that

$$f(2) - f(0) = f(c)\ (2 - 0)$$

Now $f(2) = 6$, $f(0) = 0$, and $f(x) = 3x^2 - 1$, so this equation becomes

$$6 = (3c^2 - 1)\ 2 = 6c^2 - 2$$

which gives $c^2 = 4/3$, that is, $c = \mp 2/\sqrt{3}$. But c must lie in (0, 2), so $c = 2/\sqrt{3}$. Figure 11 illustrates this calculation: The tangent line at this value of c is parallel to the secant line OB.

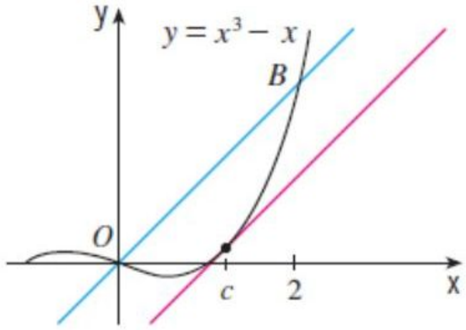


Figure 11

4.2.3 Monotonic Functions and the First Derivative Test

In sketching the graph of a differentiable function it is useful to know where it increases (rises from left to right) and where it decreases (falls from left to right) over an interval.

4.2.4 Increasing Functions and Decreasing Functions

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).
If $f(x) > 0$ at each point $x \in (a, b)$, then f is increasing on [a, b].
If $f(x) < 0$ at each point $x \in (a, b)$, then f is increasing on [a, b].
If $f(x) = 0$ at each point $x \in (a, b)$, then f is critical point (may be max. or min).

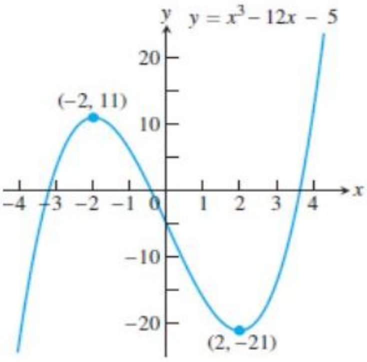
Example 9: Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which f is increasing and on which f is decreasing.

Solution: The function f is everywhere continuous and differentiable. The first derivative

$$f(x) = 3x^2 - 12 = 3(x^2 - 4)$$
$$= 3(x + 2)(x - 2)$$

is zero at $x = -2$ and $x = 2$.
These critical points subdivide the domain of f to create nonoverlapping open intervals:
$(-\infty, -2)$
$(-2, 2)$
$(2, \infty)$
on which f is either positive or negative.

We determine the sign of f by evaluating f at a convenient point in each subinterval. The behavior of f is determined by then applying the above definition to each subinterval.
The results are summarized in the following table, and the graph of f is given in Figure 12.



Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
f' evaluated	$f'(-3) = 15$	$f'(0) = -12$	$f'(3) = 15$
Sign of f'	+	−	+
Behavior of f	increasing	decreasing	increasing

Figure 12

4.2.5 First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself.
Moving across this interval from left to right,
1. if f changes from negative to positive at c, then f has a local minimum at c;
2. if f changes from positive to negative at c, then f has a local maximum at c;
3. if f does not change sign at c (that is, is positive on both sides of c or negative on both sides), then f has no local extremum at c.

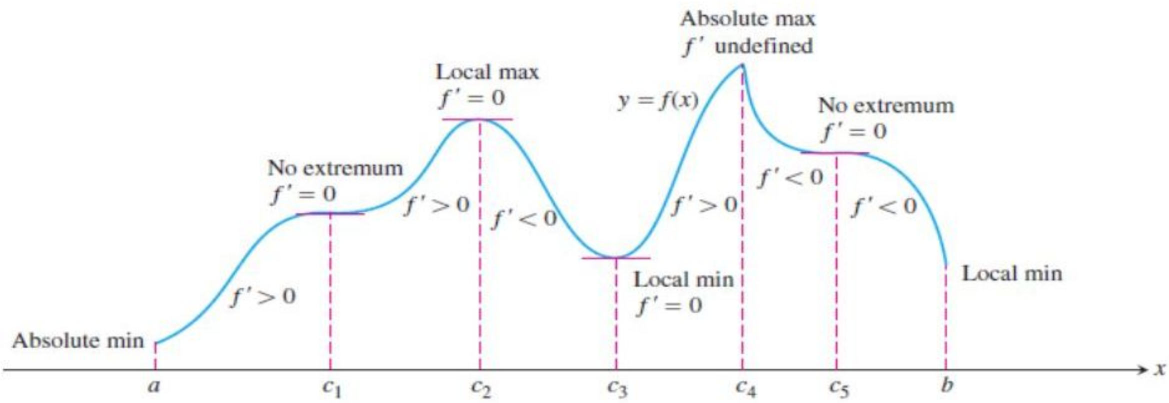


Figure 13

Example 10: Find the critical points of

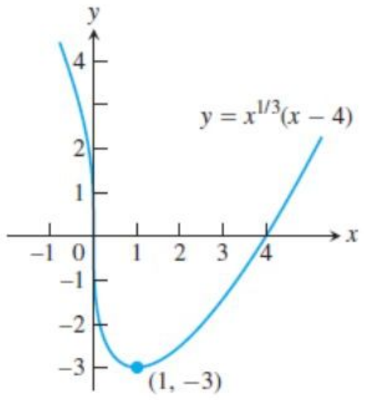
$$f(x) = x^{1/3}(x - 4) = x^{4/3} - 4x^{1/3}$$

Identify the intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution: The function f is continuous at all x since it is the product of two continuous functions, $x^{1/3}$ and $(x - 4)$. The first derivative

$$f'(x) = \frac{d}{dx}\left(x^{4/3} - 4x^{1/3}\right) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$$
$$= \frac{4}{3}x^{-2/3}\left(x - 1\right) = \frac{4(x - 1)}{3x^{2/3}}$$

is zero at $x = 1$ and undefined at $x = 0$ (see Figure 14).



Interval	$x < 0$	$0 < x < 1$	$x > 1$
Sign of f'	−	−	+
Behavior of f	decreasing	decreasing	increasing

Figure 14

4.2.6 Concavity and Curve Sketching

4.2.6a Concavity

Definition: The graph of a differentiable function $y = f(x)$ is
(a) Concave up on an open interval I if f is increasing on I;
(b) Concave down on an open interval I if f is decreasing on I.

4.2.6b The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f > 0$ on I, the graph of f over I is **concave up**.
2. If $f < 0$ on I, the graph of f over I is **concave down**.

Example11

(a) The curve (Figure 15) is concave down on $(-\infty, \infty)$ where $y = 6x < 0$ and concave up on $(0, \infty)$ where $y = 6x > 0$.

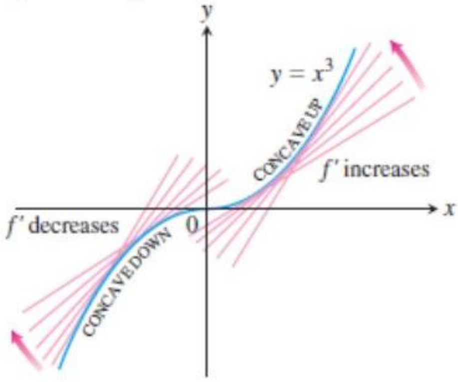


Figure 15

(b) The curve $y = x^2$ (Figure 16) is concave up on $(-\infty, \infty)$ because its second derivative $y = 2$ is always positive.

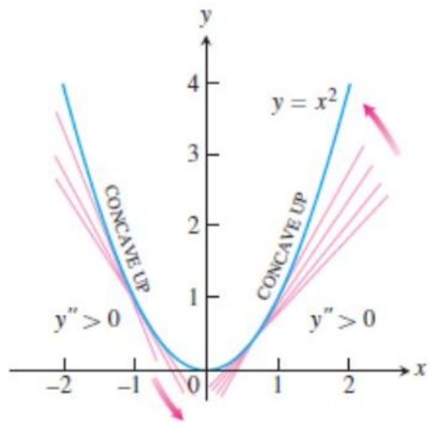


Figure 16

Example 12: Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

Solution The first derivative of $y = 3 + \sin x$ is $y = \cos x$ and the second derivative is $y = -\sin x$

The graph of $y = 3 + \sin x$ is concave down on $(0, \pi)$, where $y = -\sin x$ is negative. It is concave up on $(\pi, 2\pi)$, where $y = -\sin x$ is positive (Figure 17).

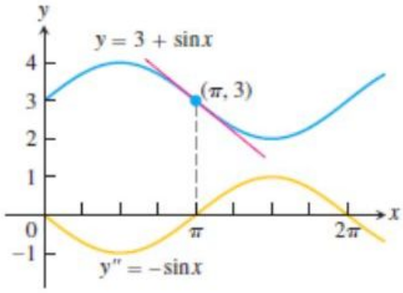


Figure 17

4.2.6c Points of Inflection

A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

Note: At a point of inflection $(c, f(c))$, either $f\ (c) = 0$ or $f(c)$ fails to exist.

Example 13: The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin because $f(x) = (5/3)\, x^{2/3} = 0$ when $x = 0$. However, the second derivative

$$f(x) = \frac{d}{dx}\left(\frac{5}{3}x^{\frac{2}{3}}\right) = \frac{10}{9}x^{-1/3}$$

fails to exist at $x = 0$. Nevertheless, $f(x) < 0$ for $x < 0$ and $f(x) > 0$ for $x > 0$, so the second derivative changes sign at $x = 0$ and there is a point of inflection at the origin. The graph is shown in Figure 18.

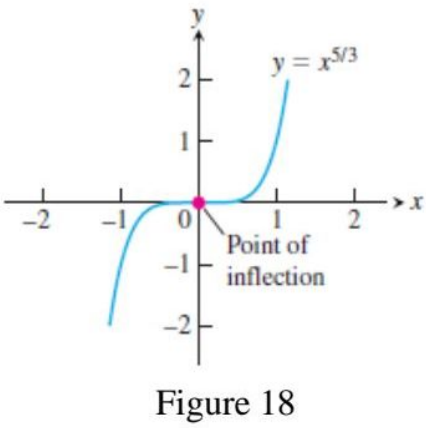


Figure 18

Example 14: The curve $y = x^4$ has no inflection point at $x = 0$ (Figure 19). Even though the second derivative $y = 12\,x^2$ is zero there, it does not change sign.

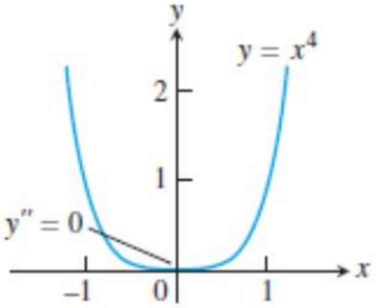


Figure 19

Example 15: The graph of $y = x^{1/3}$ has a point of inflection at the origin because the second derivative is positive for $x < 0$ and negative for $x > 0$:

$$y = \frac{d^2}{dx^2}\left(x^{\frac{1}{3}}\right) = \frac{d}{dx}\left(\frac{1}{3}x^{-\frac{2}{3}}\right) = -\frac{2}{9}x^{-5/3}$$

However, both $y = x^{-2/3}/3$ and y fail to exist at $x = 0$, and there is a vertical tangent there. See Figure 20.

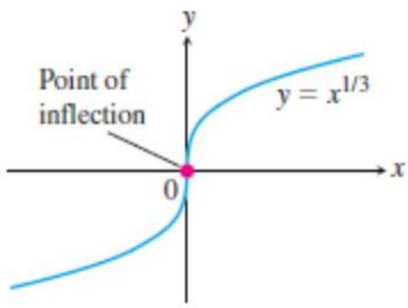


Figure 20

4.2.7 Second Derivative Test for Local Extrema

Suppose f is continuous on an open interval that contains $x = c$.

1. If $f(c) = 0$ and $f(c) < 0$, then f has a local maximum at $x = c$.

2. If $f(c) = 0$ and $f(c) > 0$, then f has a local maximum at $x = c$.

3. If $f(c) = 0$ and $f(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

4.2.8 Curve Sketching

Procedure for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find the derivatives y and y.
3. Find the critical points of f, if any, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes that may exist.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

Example 16: Sketch the curve $y = (x-2)^3 + 1$
Solution:

1. Domain = $(-\infty, \infty)$

Symmetry: $f(x) = y = x^3 - 6x^2 + 12x - 8$
$f(-x) = y = (-x)^3 - 6(-x)^2 + 12(-x) - 8$
$f(x) = y = -x^3 - 6x^2 - 12x - 8$
$f(x) = y = -(x^3 + 6x^2 - 12x + 8)$

$f(-x) \neq f(x)$ and $f(x) \neq -f(x)$
the function nor odd or even

2. $y = 3(x-2)^2(1) + 0 = 3(x-2)^2$

$y = 0 \quad 3(x-2)^2 = 0 \quad 3(x-2)(x-2) = 0 \quad x = 2$
no maximum or minimum point at $x = 2$

$y = 6(x-2) = 6x - 12$
$y = 0 \quad 6x - 12 = 0 \qquad x = 2$
at $x = 2$ inflection point
at $x = 2 \quad y = (2-2)^3 + 1 = 1$
(2, 1) is inflection point

3. Intercepts

- For x – intercept let $y = 0$

$0 = (x - 2)^3 + 1$

By inspection, $y = 0$ *if* $x = 1$

(1, 0) is the x – intercept

- For y – intercept, let $x = 0$

$y = (0 - 2)^3 + 1 = -8 + 1 = -7$

(0, -7) is the y - intercept

Example 17: sketch the curve $y = \frac{1}{4} x^4 - x^3 + 4x + 2$

Solution:

1. $y = x^3 - 3x^2 + 4 + 0$

$y = 0 \qquad x^3 - 3x^2 + 4 = 0$

if $x = -1 \qquad (-1)^3 - 3(-1)^2 + 4 = 0$

$(x + 1)(x^2 - 4x + 4) = 0$

$(x + 1) = 0 \qquad x = -1$

$x^2 - 4x + 4 = 0 \qquad (x - 2)(x - 2) = 0 \qquad x = 2$

At $x = -1$ local min. point

$y = \frac{1}{4} + 1 - 4 + 2$

$= -\frac{3}{4}$

(-1, -3/4) is min. point

2. $y = 3x^2 - 6x$

$y = 0 \qquad 3x^2 - 6x = 0 \qquad 3x(x - 2) = 0$

$3x = 0 \quad x = 0$

$x - 2 = 0 \quad x = 2$

(0, 2) (2, 6) are inflection points

3. Intercepts:

- y-intercept: $x = 0$

$y = 0 - 0 + 0 + 2 = 2$

(0, 2) is y-intercept

Example 18: Sketch $y = \sin x + \cos x$ from $x = -\pi/4$ to $3\pi/4$
Solution:
$y = \cos x - \sin x$
$y = 0 \quad \sin x = \cos x \quad \tan x = 1 \quad x = \pi/4$
$at\ x = \pi/4 \quad y = \sin \pi/4 + \cos \pi/4 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$
$(\pi/4, \sqrt{2})$ is max. point

2. $y = -\sin x - \cos x$
$-\sin x - \cos x = 0 \quad \tan x = -1 \quad x = -\pi/4 \quad and\ x = 3\pi/4$

At $x = -\pi/4 \quad y = \sin -\pi/4 + \cos -\pi/4 = \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 0$
At $x = 3\pi/4 \quad y = 0$

4. Intercepts:

$y = 0 \quad \sin x + \cos x = 0$
$\sin x = -\cos x \quad \tan x = -1$
$x = -\pi/4 \ \ and \ \ x = 3\pi/4$

y – intercept : $x = 0 \quad y = 0$

4.2.9 Sketching of Rational Functions

In graphing of rational functions, we must early know the asymptotes.
Asymptotes: if the distance between the graph and some fixed line approaches zero as the graph moves farther and farther from the origin, we say that this line is asymptotes of the graph.
There are four types of asymptotes:

1. Horizontal asymptotes
2. Vertical asymptotes
3. Oblique asymptotes
4. Curved asymptotes

1. Horizontal asymptotes:

The line $y = b$ is horizontal asymptote of the graph $y = f(x)$ if either: $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$

Example 19: Find the horizontal asymptotes for $y = 1/(x-1)$

Solution:

$\lim_{x\to\infty} 1/(x-1) = 1/\infty\text{-}1 = 1/\infty = 0$

$\lim_{x\to-\infty} 1/(x-1) = 1/\text{-}\infty\text{-}1 = 1/\text{-}\infty = 0$

$y = 0$ is horizontal asymptotes (x-axis)

Example 20: Find the horizontal asymptotes for the function $y = \frac{\sqrt{2x^2+1}}{3x-5}$

Solution:

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{\sqrt{\frac{2x^2}{x^2}+\frac{1}{x^2}}}{\frac{3x}{x}-\frac{5}{x}} = \frac{\sqrt{2+0}}{3-0} = \frac{\sqrt{2}}{3}$$

$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to-\infty} \frac{\frac{1}{x}\sqrt{2x^2+1}}{\frac{1}{x}(3x-5)} = \lim_{x\to-\infty} \frac{\frac{1}{-\sqrt{x^2}}\sqrt{2x^2+1}}{\frac{1}{x}(3x-5)}$$

Since $\sqrt{x^2} = f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

And we have $x \to -\infty$

Then $\sqrt{x^2} = -x \qquad x = -\sqrt{x^2} \qquad \frac{1}{x} = \frac{1}{-\sqrt{x^2}}$

$$\lim_{x\to-\infty} \frac{-\sqrt{\frac{2x^2}{x^2}+\frac{1}{x^2}}}{\frac{3x}{x}-\frac{5}{x}} = -\frac{\sqrt{2}}{3}$$

The horizontal asymptotes are $y = \frac{\sqrt{2}}{3}$ and $y = \frac{-\sqrt{2}}{3}$

2. Vertical asymptotes:

The line $x = a$ is vertical asymptote of the graph $y = f(x)$ if either:

$$\lim_{x\to a^+} f(x) = \pm\infty \text{ or } \lim_{x\to a^-} f(x) = \pm\infty$$

- To find vertical asymptotes, find the values of x that make the denominator equal zero and check that the limit of a function goes to (∞ or $-\infty$) as x approaches (a^+ or a^-)

Example 21: Find the vertical asymptotes for the function $y = \frac{1}{1-x}$

Solution:

$x - 1 = 0 \quad x = 1$

$\lim_{x\to 1^+} \frac{1}{x-1} = \frac{1}{0} = \infty$

$\lim_{x\to 1^-} \frac{1}{x-1} = \frac{1}{0^-} = -\infty$

$x = 1$ is vertical asymptote.

Example 22: Find the vertical asymptotes for the function $y = \frac{x^2+x-6}{x^2-4}$

Solution:

$x^2 - 4 = 0 \qquad x^2 = 4 \quad x = \pm 2$

$\lim_{x\to 2^+} \frac{x^2+x-6}{x^2-4} = \frac{0}{0} \neq \infty$ or $-\infty$

$\lim_{x\to 2^-} \frac{x^2+x-6}{x^2-4} = \frac{0}{0} \neq \infty$ or $-\infty$

$x = 2$ is not a vertical asymptote.

$\lim_{x\to -2^+} \frac{x^2+x-6}{x^2-4} = \frac{-4}{0} = \infty$

$x = -2$ is the vertical asymptote.

3. Oblique (or slant) asymptotes:

When $\lim_{x\to\pm\infty} [f(x) - (mx + b)] = 0$, then the line $y = mx + b$ is oblique asymptotes for the function $f(x)$.

- To find oblique asymptotes, divide the numerator over the denominator (by long division), the result represents the oblique asymptotes.
- If the rational function has degree of numerators is one greater than the degree of denominator, the graph has an oblique asymptotes.

Example 23: Find the oblique asymptotes of $y = \frac{x^2-3}{2x-4}$

Solution: degree of numerator – degree of denominator = 2 – 1 = 1

Use long division

$y = \frac{x}{2} + 1 + \frac{1}{2x-4}$

$y = x/2 + 1$ is the oblique asymptote.

Note: A function may have oblique asymptote but is not rational function, for example $y = \sqrt{4x^2 + 9}$ has two oblique asymptotes $y = 2x$ and $y = -2x$

4. Curved asymptotes

If the degree of numerator is more than one greater than the degree of denominator, the asymptote becomes curved.

- To find curved asymptotes, use only long division.

Example 24: Find the curved asymptotes of the function $y = \frac{x^4+1}{x^2}$

Solution:

$y = x^2 + \frac{1}{x^2}$

$y = x^2$ is the curved asymptote.

Example 25: Sketch the graph of $f(x) = \frac{(x+1)^2}{1+x^2}$

Solution

1. The domain of f is $(-\infty, \infty)$ and there are no symmetries about either axis or the origin.

2. *Find f and f*

$$f(x) = \frac{(x+1)^2}{1+x^2}$$

x – intercept at $x = -1$,

y – intercept at $(y = 1)$ at $x = 0$

$$f(x) = \frac{(1+x^2).2(x+1) - (x+1)^2.2x}{(1+x^2)^2}$$

$$= \frac{2(1-x^2)}{(1+x^2)^2}$$

Critical points: $x = -1, x = 1$

$$f(x) = \frac{(1+x^2)^2.2(-2x) - 2(1-x^2)[2(1+x^2).2x]}{(1+x^2)^4}$$

$$= \frac{4x(x^2-3)}{(1+x^2)^3}$$

3. *Behavior at critical points*. The critical points occur only at $x = \pm 1$ where $f(x) = 0$ (Step 2) since f exists everywhere over the domain of f. At $x = -1$, $f(-1) = 1 > 0$ yielding a relative minimum by the Second Derivative Test. At $x = 1$, $f(1) = -1 < 0$ yielding a relative maximum by the Second Derivative test.

4. *Increasing and decreasing.* We see that on the interval $(-\infty, -1)$ the derivative $f(x) < 0$, and the curve is decreasing. On the interval $(-1, 1)$, $f(x) > 0$ and the curve is increasing; it is decreasing on $(1, \infty)$ where $f(x) < 0$ again.

5. *Inflection points*. Notice that the denominator of the second derivative (Step 2) is always positive. The second derivative f is zero when $x = -\sqrt{3}$, 0 and $\sqrt{3}$. The second derivative changes sign at each of these points: negative on $(-\infty, -\sqrt{3})$, positive on $(\sqrt{3}, 0)$, negative on $(0, \sqrt{3})$, and positive again on $(\sqrt{3}, \infty)$. Thus each point is a point of inflection. The curve is concave down on the interval $(-\infty, -\sqrt{3})$ concave up on $(-\sqrt{3}, 0)$, concave down on $(0, \sqrt{3})$ and concave up again on $(\sqrt{3}, \infty)$.

6. *Asymptotes.* Expanding the numerator of $f(x)$ and then dividing both numerator and denominator by x^2 gives

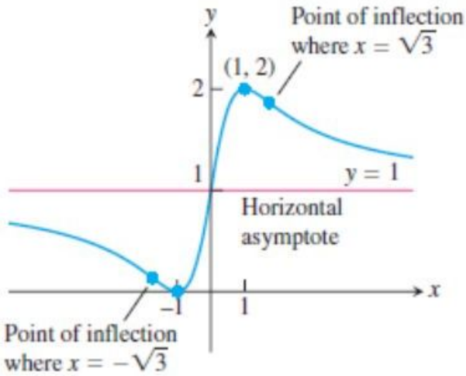
$$f(x) = \frac{(x+1)^2}{1+x^2} = \frac{x^2+2x+1}{1+x^2}$$

$$\frac{1+\left(\frac{2}{x}\right)+\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)+1}$$

We see that $f(x) \rightarrow 1^+$ as $x \rightarrow \infty$ and that $f(x) \rightarrow 1^-$ as $x \rightarrow -\infty$. Thus, the line $y = 1$ is a horizontal asymptote.

Since f decreases on $(-\infty, -1)$ and then increases on $(-1, 1)$ we know that $f(-1) = 0$ is a local minimum. Although f decreases on $(1, \infty)$ it never crosses the horizontal asymptote $y = 1$ on that interval (it approaches the asymptote from above). So the graph never becomes negative, and $f(-1) = 0$ is an absolute minimum as well. Likewise, $f(1) = 2$ is an absolute maximum because the graph never crosses the asymptote $y = 1$ on the interval $(-\infty, -1)$ approaching it from below. Therefore, there are no vertical asymptotes (the range of f is $(0 \le y \le 2)$.

7. The graph of f is sketched in Figure. Notice how the graph is concave down as it approaches the horizontal asymptote $y = 1$ as $x \to -\infty$ and concave up in its approach to $y = 1$ as $x \to \infty$



Example 26: Sketch the graph of $f(x) = \frac{x^2+4}{2x}$

Solution:

1. The domain of f is all nonzero real numbers. There are no intercepts because neither x nor $f(x)$ can be zero. Since $f(-x) = -f(x)$, we note that f is an odd function, so the graph of f is symmetric about the origin.

2. We calculate the derivatives of the function, but first rewrite it in order to simplify our computations:

$$f(x) = \frac{x^2 + 4}{2x} = \frac{x}{2} + \frac{2}{x}$$

$$f(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2 - 4}{2x^2}$$

$$f(x) = \frac{4}{x^3}$$

3. The critical points occur at $x = \pm 2$ where $f(x) = 0$. Since $f(-2) < 0$ and $f(2) > 0$, we see from the Second Derivative Test that a relative maximum occurs at $x = -2$ with $f(-2) = -2$, and a relative minimum occurs at $x = 2$ with $f(2) = 2$.

4. On the interval $(-\infty, -2)$ the derivative f is positive because $x^2 - 4 > 0$ so the graph is increasing; on the interval $(-2, 0)$ the derivative is negative and the graph is decreasing. Similarly, the graph is decreasing on the interval $(0, 2)$ and increasing on $(2, \infty)$.

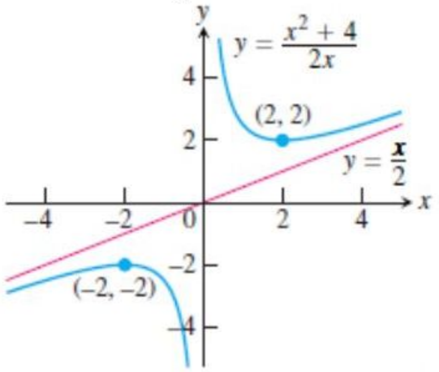
5. There are no points of inflection because $f(x) < 0$ whenever $x < 0$, $f(x) > 0$ whenever $x > 0$and f exists everywhere and is never zero throughout the domain of f. The graph is concave down on the interval $(-\infty, -2)$ and concave up on the interval $(0, \infty)$.

6. From the rewritten formula for $f(x)$, we see that

$$\lim_{x\to 0^+}\left(\frac{x}{2}+\frac{2}{x}\right) = +\infty \text{ and } \lim_{x\to 0^-}\left(\frac{x}{2}+\frac{2}{x}\right) = -\infty,$$

so the y-axis is a vertical asymptote. Also, as $x\to\infty$ or as $x\to -\infty$ the graph of $f(x)$ approaches the line $y = x/2$. Thus is an oblique asymptote.

7. The graph of f is sketched in Figure below



Example 27: Use symmetry, first derivative, second derivative, and asymptotes to graph the function $y = \frac{x^2}{x^2-1}$

Solution:

1. Symmetry:

$$f(-x) = \frac{(-x)^2}{(-x)^2-1} = \frac{x^2}{x^2-1} = f(x)$$

The function is even function (symmetric about y-axis)

2. First derivative:

$$y = 1 + \frac{1}{x^2 - 1}$$

$$y = 0 - \frac{2x}{(x^2 - 1)^2}$$

$$y = \frac{-2x}{(x^2-1)^2} = 2x(\frac{-1}{(x^2-1)^2})$$

$y = 0$ $\quad \frac{-1}{(x^2-1)^2} \neq 0$

$2x = 0 \quad x = 0$

$x^2 - 1 = 0 \quad x^2 = 1 \quad x = \pm 1$ [at these values y is not defined]

at $x = 0$ $\quad y = 1 + \frac{1}{(0)^2-1} = 1 - 1 = 0$

(0, 0) is maximum point.

3. Second derivative:

$$y = \frac{-2x}{(x^2-1)^2} \qquad y = \frac{(x^2-1)^2(-2)-(-2x)[2(x^2-1)(2x)]}{(x^2-1)^4}$$

$$y = \frac{-2(x^2-1)^2+8x^2(x^2-1)}{(x^2-1)^4} = \frac{(x^2-1)[-2(x^2-1)+8x^2]}{(x^2-1)^4}$$

$$y = \frac{6x^2+2}{(x^2-1)^3} = (6x^2 + 2)(\frac{1}{(x^2-1)^3})$$

$y = 0$ $\quad \frac{1}{(x^2-1)^3} \neq 0,$

$6x^2 + 2 = 0 \quad x^2 \neq -1/3 \quad x \neq \sqrt{-\frac{1}{3}}$

$y \neq 0$

The values of x that make y not defined:

$x^2 - 1 = 0 \quad x^2 = 1 \quad x = \pm 1$

$x = -1, x = 1$ are out of domain

No inflection points

4. Asymptotes:

Horizontal asymptotes:

$\lim_{x\to\infty} 1+\frac{1}{x^2-1} = 1 + 1/\infty = 1 + 0 = 1$

$\lim_{x\to-\infty} 1+\frac{1}{x^2-1} = 1 + 1/- \infty = 1 + 0 = 1$

$y = 1$ is horizontal asymptote.

Vertical asymptotes:

$x^2 - 1 = 0 \qquad x^2 = 1 \qquad x = \pm 1$

$\lim_{x\to1^+} 1+\frac{1}{x^2-1} = 1 + \infty = \infty$

$\lim_{x\to-1^+} 1+\frac{1}{x^2-1} = 1 - \infty = \infty$

$x = 1$ and $x = - 1$ are vertical asymptotes.

Example 28: use first derivative, second derivative and the asymptotes to graph

$y = \frac{x^2-4}{x-1}$

Solution:

1. Asymptotes:

$y = x + 1 - \frac{3}{x-1}$

$y = x + 1$ is oblique asymptote.

- Horizontal asymptote:

$\lim_{x\to\infty} \frac{x^2-4}{x-1} = \frac{\infty}{\infty}$ (indeterminate form)

No horizontal asymptote

- Vertical asymptote:

$x - 1 = 0 \qquad x = 1$

$\lim_{x\to1^+} x + 1 - 3/(x-1) = - \infty$

$x = 1$ is vertical asymptote

2. First derivative:

$y = 1 + 0 + \frac{3}{(x-1)^2}$

$= \frac{3}{(x-1)^2} + 1$

$y = 0 \qquad \frac{3}{(x-1)^2} = -1 \qquad (x-1)^2 = -3 \qquad x^2 - 2x + 4 = 0$

$$x = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

$x = \frac{-(-2) \mp \sqrt{(-2)^2 - 4(1)(4)}}{2(1)} = \frac{2 \mp \sqrt{-12}}{2}$ (not defined)

$y \neq 0$ no max. or min. point

The value of x that makes y not defined is $x - 1 = 0 \qquad x = 1$

3. Second derivative:

$$y = 1 + \frac{3}{(x-1)^2}$$

$$y = 0 + \frac{[(x-1)^2 * 0] - [3 * 2(x-1)(1)]}{(x-1)^4}$$

$$y = \frac{-6(x-1)}{(x-1)^4} = \frac{-6}{(x-1)^3}$$

$y = 0 \qquad x = 1$ (y not defined)

$x = 1$ out of domain no inflection point

4.3 L'Hôpital's Rule

4.3.1 Indeterminate Forms 0/0

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g(x) \neq 0$ on I if $x \neq a$.

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x)}{g(x)}$$

assuming that the limit on the right side of this equation exists.

Example 29: The following limits involve 0/0 indeterminate forms, so we apply l'Hôpital's Rule. In some cases, it must be applied repeatedly.
Solution:

(a) $\lim_{x\to 0}\frac{3x-\sin x}{x}=\lim_{x\to 0}\frac{3-\cos x}{1}=\frac{3-\cos x}{1}\bigg|_{x=0}=2$

(b) $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}}{1}=\frac{1}{2}$

(c) $\lim_{x\to 0}\frac{\sqrt{1+x}-1-x/2}{x^2}$ $\qquad \frac{0}{0}$

$=\lim_{x\to 0}\frac{(1/2)(1+x)^{-1/2}-1/2}{2x}$ $\qquad$ Still $\frac{0}{0}$; differentiate again.

$=\lim_{x\to 0}\frac{-(1/4)(1+x)^{-3/2}}{2}=-\frac{1}{8}$ $\qquad$ Not $\frac{0}{0}$; limit is found.

(d) $\lim_{x\to 0}\frac{x-\sin x}{x^3}$ $\qquad \frac{0}{0}$

$=\lim_{x\to 0}\frac{1-\cos x}{3x^2}$ $\qquad$ Still $\frac{0}{0}$

$=\lim_{x\to 0}\frac{\sin x}{6x}$ $\qquad$ Still $\frac{0}{0}$

$=\lim_{x\to 0}\frac{\cos x}{6}=\frac{1}{6}$ $\qquad$ Not $\frac{0}{0}$; limit is found.

Using L'Hôpital's Rule
To find

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, continue to differentiate f and g, so long as we still get the form 0/0 at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$, we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

Example 30: Evaluate the following limits:

a) $\lim_{x\to 0}\frac{1-\cos x}{x+x^2}$, b) $\lim_{x\to 1}\frac{x^2-4x+3}{x-1}$

Solution:

a) $\lim_{x\to 0}\frac{1-\cos x}{x+x^2}=\frac{0}{0}$

$$=\lim_{x\to 0}\frac{\sin x}{1+2x}=\frac{0}{1}=0$$

Note: now if we want to find $\lim_{x\to 0}\frac{\sin x}{1+2x}$ by applying l'Hôpital's Rule:

$$=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}$$

which is **not the correct** limit. L'Hôpital's Rule can only be applied to limits that give **indeterminate forms**, and 0/1 is not an indeterminate form.

b) $\lim_{x\to 1}\frac{x^2-4x+3}{x-1}$

$$\lim_{x\to 1}\frac{x^2-4x+3}{x-1}=\frac{0}{0}$$

$$=\lim_{x\to 1}\frac{2x-4}{1}=\frac{2(1)-4}{1}=-2$$

Note: Sometimes when we try to evaluate a limit as $x \to a$ by substituting $x = a$ we get an indeterminant form like ∞/∞, ∞.0, or ∞ - ∞, instead of 0/0. We will consider the following form:

4.3.2 Indeterminate Forms ∞/∞

It is proved that l'Hôpital's Rule applies to the indeterminate form ∞/∞ as well as to 0/0 as shown in following examples:

Example 31: Find the limits of $\lim_{x\to \pi/2}\frac{\sec x}{1+\tan x}$

Solution:

$$\lim_{x\to\pi/2} \frac{secx}{1+tanx} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\pi/2} \frac{secxtanx}{sec^2x}$$

$$= \lim_{x\to\pi/2} sinx = 1$$

Example 32: Find $\lim_{x\to\infty} \frac{2x^3+3x^2+1}{x^2+4}$

Solution:

$$\lim_{x\to\infty} \frac{2x^3+3x^2+1}{x^2+4} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{6x^2+6x}{2x}$$

$$= \lim_{x\to\infty} \frac{2x(3x+3)}{2x}$$

$$= \lim_{x\to\infty} 3x+3 = \infty$$

4.3.3 Indeterminate Forms ∞.0, ∞ − ∞

Sometimes these forms can be handled by using algebra to convert them to a 0/0 or ∞/∞

Example 33: Find the limits of $\lim_{x\to\infty}(x \sin\frac{1}{x})$

Solution:

$$\lim_{x\to\infty}(x \sin\frac{1}{x}) = 0.\infty$$

Let *h = 1/x*:

$$\lim_{x\to\infty}(x \sin\frac{1}{x}) = \lim_{h\to 0}\left(\frac{1}{h}\sin h\right) = \lim_{h\to\infty}\frac{sinh}{h} = 1$$

Example 34: Find the limit of this ∞ - ∞ form:

$$\lim_{x\to 0}\left(\frac{1}{sinx}-\frac{1}{x}\right)$$

Solution:

$$\lim_{x\to 0}\left(\frac{1}{sinx}-\frac{1}{x}\right)= \lim_{x\to 0}\frac{1}{sinx}-\lim_{x\to 0}\frac{1}{x}= \infty-\infty$$

$$\lim_{x\to 0}\left(\frac{1}{sinx}-\frac{1}{x}\right)=\lim_{x\to 0}\frac{x-sinx}{xsinx}$$

$$=\lim_{x\to 0}\frac{1-cosx}{sinx+xcosx}$$

Still 0/0

Use L'Hôpital's Rule again:

$$=\lim_{x\to 0}\frac{sinx}{2cosx-xsinx}=\frac{0}{2}=0$$

4.4 Applied Optimization

In this section we use derivatives to solve a variety of optimization problems in business, mathematics, physics, and economics.

Solving Applied Optimization Problems

1. *Read the problem*. Read the problem until you understand it. What is given? What is the unknown quantity to be optimized?
2. *Draw a picture*. Label any part that may be important to the problem.
3. *Introduce variables*. List every relation in the picture and in the problem as an equation or algebraic expression, and identify the unknown variable.

4. *Write an equation for the unknown quantity*. If you can, express the unknown as a function of a single variable or in two equations in two unknowns. This may require considerable manipulation.
5. *Test the critical points and endpoints in the domain of the unknown*. Use what you know about the shape of the function's graph. Use the first and second derivatives to identify and classify the function's critical points.

Example 35: An open-top box is to be made by cutting small congruent squares from the corners of a 12-in.-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

Solution:
We start with a picture (Figure 23).

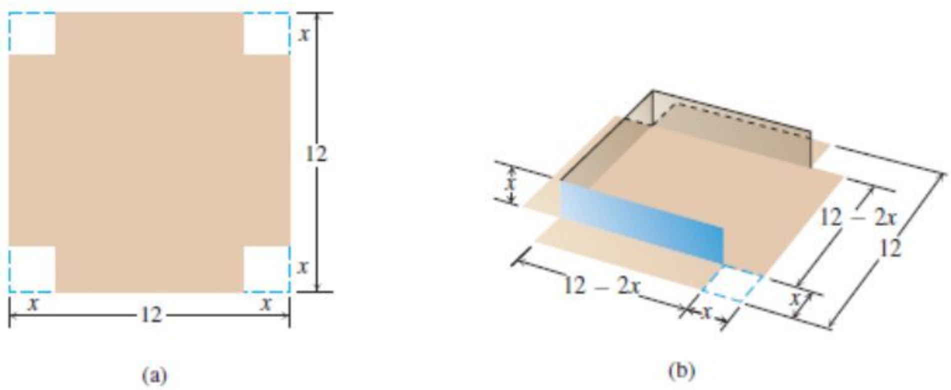


Figure 23

In the figure, the corner squares are x in. on a side. The volume of the box is a function of this variable:

$$V(x) = x(12 - 2x)^2 = 144x - 48x^2 + 4x^3$$

Since the sides of the sheet of tin are only 12 in. long, $x \le 6$ and the domain of V is the interval $0 \le x \le 6$.

A graph of V (Figure 24) suggests a minimum value of 0 at $x = 0$ and $x = 6$ and a maximum near $x = 2$.To learn more, we examine the first derivative of V with respect to x:

$$dV/dx = 144 - 96x + 12x^2 = 12(12 - 8x + x^2) = 12(2 - x)(6 - x)$$

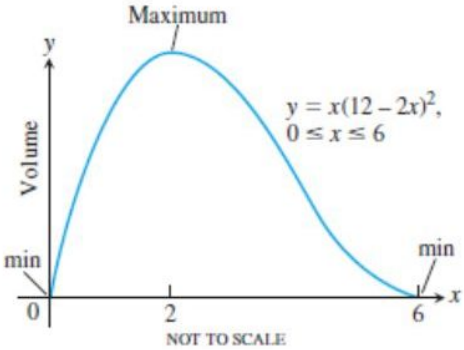


Figure 24

Of the two zeros, $x = 2$ and $x = 6$, only $x = 2$ lies in the interior of the function's domain and makes the critical-point list. The values of V at this one critical point and two endpoints are:

Critical-point value: $V(2) = 128$
Endpoint values: $V(0) = 0$, $V(6) = 0$.

The maximum volume is 128 in^3. The cutout squares should be 2 in. on a side.

Example 36: You have been asked to design a one-liter can shaped like a right circular cylinder (Figure 25). What dimensions will use the least material?
Solution:

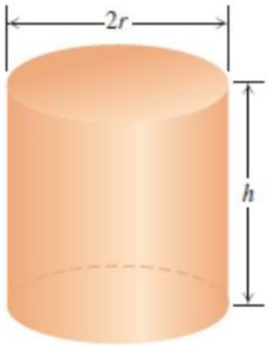


Figure 25

Volume of can: If r and h are measured in centimeters, then the volume of the can in cubic centimeters is

$$\pi r^2 h = 1000 \qquad [1 \text{ liter} = 1000 \text{ cm}^3]$$

Surface area of can:

$$A = \underset{\substack{\text{circular}\\ \text{end}}}{2\pi r^2} + \underset{\substack{\text{cylindrical}\\ \text{wall}}}{2\pi rh}$$

How can we interpret the phrase "least material"? For a first approximation we can ignore the thickness of the material and the waste in manufacturing. Then we ask for dimensions r and h that make the total surface area as small as possible while satisfying the constraint $\pi r^2 h = 1000$.

To express the surface area as a function of one variable, we solve for one of the variables in $\pi r^2 h = 1000$ and substitute that expression into the surface area formula. Solving for h is easier:

$$h = \frac{1000}{\pi r^2}$$

Thus,

$$\begin{aligned} A &= 2\pi r^2 + 2\pi rh \\ &= 2\pi r^2 + 2\pi r \left(\frac{1000}{\pi r^2}\right) \\ &= 2\pi r^2 + \left(\frac{2000}{r}\right) \end{aligned}$$

Our goal is to find a value of $r > 0$ that minimizes the value of A. Figure 26 suggests that such a value exists.

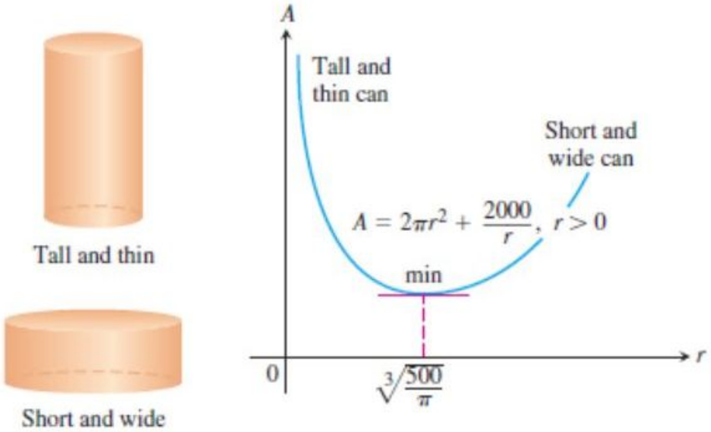


Figure 26

Notice from the graph that for small r (a tall, thin cylindrical container), the term $2000/r$ dominates and A is large. For large r (a short, wide cylindrical container), the term $2\pi r^2$ dominates and A again is large.

Since A is differentiable on $r > 0$, an interval with no endpoints, it can have a minimum value only where its first derivative is zero.

$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2}$$

$$0 = 4\pi r - 2000/r^2$$

$$4\pi r^3 = 2000$$

$$r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42$$

The second derivative

$$\frac{d^2A}{dr^2} = 4\pi + \frac{4000}{r^3}$$

is positive throughout the domain of A. The graph is therefore everywhere concave up and the value of A at $r = \sqrt[3]{500/\pi}$ is an absolute minimum.

The corresponding value of h (after a little algebra) is

$$h = \frac{1000}{\pi r^2} = 2\sqrt[3]{\frac{500}{\pi}} = 2r$$

The one-liter can that uses the least material has height equal to twice the radius, here with $r \approx 5.42$ cm and $h \approx 10.84$ cm.

Example 37: Find the area of the largest rectangle with lower base on the x-axis and upper vertices on the parabola $y = 12 - x^2$.

Solution:

$A = 2x\,(12 - x^2)$

$A = 24x - 2x^3 \qquad 0 \le x \le 2\sqrt{3}$

$dA/dx = 24 - 6x^2$

At maximum or minimum points, $dA/dx = 0$

$24 - 6x^2 = 0$

$x^2 = 24/6 = 4$

$x = 2$ or $x = -2$ [$x = -2$ is neglected]

$d^2A/dx^2 = -12x$

At $x = 2 \qquad d^2A/dx^2 = -12 * 2 = -24 = -ve$

$x = 2$ is maximum point

Check bound:

At $x = 0 \qquad A = 0$

At $x = 2\sqrt{3} \qquad A = 0$

At $x = 2$ Absolute max.

$A = 2(2)\,[12 - (2)^2] = 32$ unit

Example 38: The height of an object moving vertically is given by $S = -16t^2 + 96t + 112$ when s in feet and t in seconds. Find:

a. The velocity when $t = 0$
b. Its maximum height
c. Its velocity when $s = 0$

Solution:

a. $Velocity = ds/dt = -32t + 96$

At $t = 0$ $v = -32(0) + 96 = 96$ ft/sec

b. At maximum height, velocity $v = 0$

$-32t + 96 = 0$

$t = 96/32 = 3$ sec

$S_{max} = -16(3)^2 + 96(3) + 112 = 256$ ft

c. At $S = 0$ $-16t^2 + 96t + 112 = 0$ $-t^2 + 6t + 7 = 0$

$t^2 - 6t - 7 = 0$

$(t-7)(t+1) = 0$

$t = 7$

$t = -1$ (neglected)

$v = -32t + 96 = -32(7) + 96 = -128$ ft/sec

Example 39: what is the smallest perimeter possible for a rectangle of area equal to 16 cm^2.

Solution:

$P = 2(x + y)$

$A = xy$

$16 = xy$ $y = 16/x$

$P = 2(x + 16/x) = 2x + 32/x$ $0 < x < \infty$

$$\frac{dp}{dx} = 2 - \frac{32}{x^2} = \frac{2x^2 - 32}{x^2}$$

$\frac{dp}{dx} = 0$ $\frac{2x^2-32}{x^2} = 0$ $2x^2 - 32 = 0$ $x^2 = 16$ $x = \pm 4$

$x = 4$ [$x = -4$ was neglect]

$$\frac{d^2p}{dx^2} = 0 - \frac{-32(2x)}{x^4} = \frac{64}{x^3}$$

At $x = 4$ $\frac{d^2p}{dx^2} = +ve$

$x = 4$ local min. point

Bound check:

At $x = 0$ $P = \infty$

At $x = \infty$ $P = \infty$

$x = 4$ give absolute min., $y = 16/4 = 4$

$P = 2(4 + 4) = 16$ cm (rectangle is square)

Example 40: The wall shown is 8ft height and stands 27 ft from the building. What is the length of the shortest straight beam that will reach to the side of building from the ground outside the wall?

Solution:

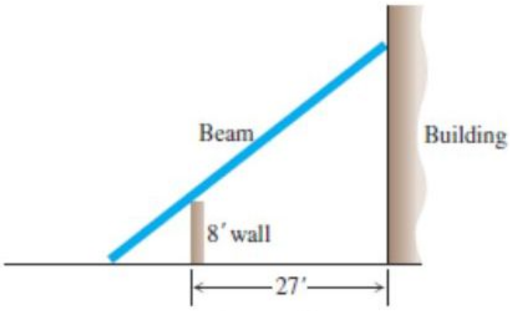


Figure 27

Let L = length of the beam

$L^2 = y^2 + (x + 27)^2$

From similar triangle:

$\frac{y}{27+x} = \frac{8}{x}$ $\quad y\,x = 8(27 + x)$ $\quad y = \frac{8(27+x)}{x}$

$L^2 = [\frac{8(27+x)}{x}]^2 + (x + 27)^2$

$L = [(\frac{8(27+x)}{x})^2 + (x + 27)^2]^{1/2}$ $\quad 0 < x < \infty$

For min. L: $dL/dx = 0$

$$\frac{dL}{dx} = \frac{1}{2}[(\frac{8(27+x)}{x})^2 + (x + 27)^2]^{-\frac{1}{2}} \times [2\left(\frac{8(27+x)}{x}\right) * \frac{8x-216-8x}{x^2} + 2(x + 27) * 1]$$

$$\frac{dL}{dx} = \frac{2[\frac{216+8x}{x} * \left(\frac{-216}{x^2}\right) + (x + 27)]}{2[\left(\frac{8(27+x)}{x}\right)^2 + (x + 27)^2]^{1/2}}$$

$$\frac{dL}{dx} = \frac{\frac{-46656 - 1728x}{x^3} + \frac{(x+27)}{1}}{2[\left(\frac{8(27+x)}{x}\right)^2 + (x+27)^2]^{1/2}}$$

$$\frac{dL}{dx} = \frac{\frac{-46656 - 1728x + x^4 + 27x^3}{x^3}}{2[\left(\frac{8(27+x)}{x}\right)^2 + (x+27)^2]^{1/2}}$$

$$\frac{dL}{dx} = 0$$

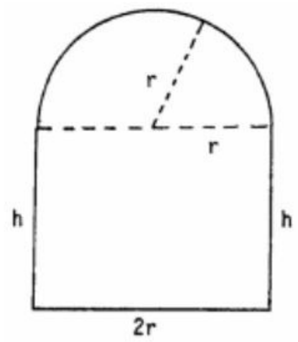
$$\frac{-46656 - 1728x + x^4 + 27x^3}{x^3} = 0$$

$$x^4 + 27x^3 - 1728x - 46656 = 0$$

By trail an error $x = 12$ ft
At $x = 12$ local min.

Bound check:
At $x = 0$ L = ∞
At $x = \infty$ L = ∞
$x = 12$ ft give absolute minimum of L

Example 41: A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass, whereas the semicircle is of tinted glass that transmits only half as much light per unit area as clear glass does. The total perimeter is fixed. Find the proportions of the window that will admit the most light. Neglect the thickness of the frame.
Solution:
From the diagram:



The perimeter is: $P = 2r + 2h + \pi r$
where,
r = radius of semicircle
h = the height of the triangle
The amount of light transmitted proportional to:
$A = 2rh + (1/4)\pi r^2$
$A = r(P - 2r - \pi r) + (1/4)\pi r^2$
$= rP - 2r^2 - (3/4)\pi r^2$
$dA/dr = P - 4r - (3/2)\pi r$

$P - 4r - (3/2)\pi r = 0$
$r = 2P/(8 + 3\pi)$
$2h = P - 4P/(8 + 3\pi) - 2\pi P/(8 + 3\pi)$
$= (4 + \pi)P/(8 + 3\pi)$
Therefore, $2r/h = 8/(4 + \pi)$ gives the proportions that admit the <u>most</u> light since $d^2A/dr^2 = -4 - 3\pi/2 < 0$